ON COMPLETE LOCALLY EUCLIDEAN SPACES

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A complete locally euclidean space is a complete riemannian manifold with curvature and torsion equal to zero. The central result of this paper is

THEOREM A. *The fundamental group π of a complete locally euclidean space can be realized as the fundamental group of a compact locally euclidean space.*

In [**4**] this result was proved by using Theorem 3 of [**1**]. We now give a new proof, independent of the results of [**1**]. The present proof has the advantage of being constructive, i.e. we actually show how to construct a certain subgroup of π which is required in the proof of Theorem A. The proofs of Wolf [**5**] are different from both [**4**] and the present proofs. The main results of this paper were obtained by J. A. Wolf and the author independently, see [**5**].

If a group G of automorphisms of an n-dimensional real vector space leaves complementary subspaces A and B invariant, then G restricted to A and B shall be denoted by $p(G)$ and $q(G)$respectively. We shall write $G=p(G)+q(G)$.

THEOREM 1. *Let π be the fundamental group of a complete locally euclidean space of dimension n. Then π is finitely generated and torsion free, and contains a unique maximal abelian normal subgroup of finite index in π.*

PROOF. From the work done in [**3**], we can always choose a basis of the n-dimensional euclidean space E^n so that all of the following conditions are satisfied:

1. $\pi\subset E^n\cdot O(n)$, the semidirect product of E^n and the orthogonal group $O(n)$.
2. $\pi=p(\pi)+q(\pi)$.
3. $q(\pi)$ is a discrete uniform subgroup of $E^m\cdot O(m)$, $m\leq n$.
4. There is a group epimorphism

$$\beta\colon q(\pi)\to p(\pi)$$

defined by $\beta[q(x)]=p(x)$ for every x in π.

Since $q(\pi)$ is finitely generated and has a subgroup of finite index consisting of pure translations, it follows (easily) that π is finitely

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generated and has a normal abelian subgroup Δ of finite index generated by $\beta(T_i)+T_i$, $i=1, 2, \cdots, m$, where T_i are the maximal linearly independent translations in $q(\pi)$. π is torsion free is standard.

To prove uniqueness, construct a strictly ascending chain of normal abelian subgroups, starting with Δ. Since the subgroup generated by all the members of this chain is finitely generated, the chain must break off after finite number of steps. Hence the result.

Since the fundamental group of a compact locally euclidean space was characterized in [2], the above result implies Theorem A.

It follows from the construction of Δ in the proof of Theorem 1 that the subgroup of Δ consisting of proper euclidean motions is a fundamental group of a locally euclidean space, homeomorphic to the direct product of a euclidean space and a torus.

In conclusion we note that if the fundamental group π of a complete locally euclidean space is nilpotent, then it must be abelian. For, π can be considered to be the fundamental group of a compact locally euclidean space. Hence the homogeneous part of every element of π is unipotent and therefore cannot be of finite order. But this is not possible, because π has a normal abelian subgroup of finite index.

References

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