INEQUALITIES RELATED TO LIDSKIĬ'S[1]

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Introduction. The purpose of this paper is to present brief proofs of many of the known linear inequalities involving the eigenvalues of the sum of two hermitian matrices [1], [3], [6], [8] and [9]. We derive quite easy proofs of the inequalities due to Lidskiĭ [6] and [8] to Amir-Moéz [1] and to J. Hersch [3] and B. P. Zwahlen [9]. Proofs of Lidskiĭ's inequality appear in [4], [9] and [10]. Our interest in this very old problem arose from a study of a recent result of F. John [5]. Lidskiĭ's inequality immediately implies this result of F. John. In fact, a slight strengthening of F. John's result is (trivially) *equivalent* to the inequality of Lidskiĭ.

1. Conservation of eigenvalues and a proof of Lidskiĭ's inequality. Let $\Sigma(X)$ be the set of all hermitian operators defined on a unitary space X of dimension N. For A in $\Sigma(X)$, let $\alpha(A)=(\alpha_1, \cdots, \alpha_N)=(\alpha(1), \cdots, \alpha(N))$ be the eigenvalues of A arranged in descending order. If also B is in $\Sigma(X)$, we write $\alpha(B)=(\beta_1, \cdots, \beta_N)$, $\alpha(A+B)=(\gamma_1, \cdots, \gamma_N)$ and $e_1, \cdots, e_N; f_1, \cdots, f_N; g_1, \cdots, g_N$ for orthonormal subsets of X such that $Ae_i=\alpha_i e_i$, $Bf_i=\beta_i f_i$ and $(A+B)g_i=\gamma_i g_i$ for $1 \leqq i \leqq N$.

THEOREM 1 (WIELANDT). *If* $1 \leqq i_1 < i_2 < \cdots < i_n \leqq N$, *then*

$$\sum_{j=1}^{n} \gamma(i_j) \leqq \sum_{j=1}^{n} \alpha(i_j) + \sum_{j=1}^{n} \beta_j. \tag{Λ}$$

By use of Muirhead's theorem one sees (cf. [8, p. 110]) that Theorem 1 is equivalent to

(L) LIDSKIĬ'S INEQUALITY. *The vector* $\alpha(A+B)$ *is in the convex hull of the set* $\{\alpha(A)+\alpha(B)P;\ P \text{ in } S_N\}$, *where* S_N *is the set of all* N *by* N *permutation matrices.*

We base our proof of Theorem 1 on the following well-known

CONSERVATION PRINCIPLE. *Let* A *be in* $\Sigma(X)$ *and let* $\tilde{X}$ *be a subspace of* X *of dimension* $N-1$. *Define* $\tilde{A}$ *in* $\Sigma(\tilde{X})$ *by requiring that* $(\tilde{A}y, y)=(Ay, y)$ *for* y *in* $\tilde{X}$. *Write* $\alpha(\tilde{A})=(\tilde{\alpha}_1, \cdots, \tilde{\alpha}(N-1))$. *Then*

(1) $\alpha_1 \geqq \tilde{\alpha}_1 \geqq \alpha_2 \geqq \cdots \geqq \tilde{\alpha}(N-1) \geqq \alpha(N)$.

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(2) *If* $\tilde{X}$ *contains* $e_1, \cdots, e_k$, *then* $\tilde{\alpha}_j = \alpha_j$ *for* $1 \leqq j \leqq k$.

(3) *If* $\tilde{X}$ *contains* $e_k, \cdots, e_N$, *then* $\alpha_j = \tilde{\alpha}_{j-1}$ *for* $k \leqq j \leqq N$.

PROOF OF THEOREM 1. We use induction on N, noting that the theorem is obvious if $N=1$ or if $n=N$ so we may assume that $1, n<N$.

Case 1. $i_n<N$. Let $\tilde{X}$ be the subspace spanned by $g_1, \cdots, g_{N-1}$. Let $\tilde{A}$ and $\tilde{B}$ be the restrictions of A and B to $\tilde{X}$, respectively. By induction

$$\sum_{j=1}^{n} \tilde{\gamma}(i_j) \leqq \sum_{j=1}^{n} \tilde{\alpha}(i_j) + \sum_{j=1}^{n} \tilde{\beta}_j.$$

The inequality (Λ) follows from (1) and (2) of the conservation principle.

Case 2. $1<i_1$. Let $\tilde{X}$ be the subspace spanned by $e_2, \cdots, e_N$. Let $\tilde{A}$ and $\tilde{B}$ be defined as in Case 1. By induction

$$\sum_{j=1}^{n} \tilde{\gamma}(i_j - 1) \leqq \sum_{j=1}^{n} \tilde{\alpha}(i_j - 1) + \sum_{j=1}^{n} \tilde{\beta}_j.$$

The inequality (Λ) follows from (1) and (3) of the conservation principle.

Case 3. $i_1=1$. Let $S=\{i_1, \cdots, i_n\}$ and let

$$T = \{N+1-i;\, i \notin S,\, 1 \leqq i \leqq N\}.$$

Then $N \notin T$ and by Case 1

$$\sum (\gamma_i;\, i \in T) \leqq \sum (\alpha_i;\, i \notin T) + \sum_{j=1}^{N-n} \beta_j.$$

Apply this inequality to $-A$, $-B$, reverse the sense, add $\sum(\gamma_i; i \in S)$ to both sides and use the additivity of the trace to get the inequality (Λ).

We turn now to the recent results of F. John [**5**]. Let R_N be the set of all 1 by N real vectors. For $\sigma \subset R_N$ and Q in S_N, define

$$E(\sigma, Q) = \{A \text{ in } \Sigma(X);\, \alpha(A)Q \text{ in } \sigma\},$$

and, with F. John,

$$C(\sigma) = \bigcap(E(\sigma, Q);\, Q \in S_N), \qquad D(\sigma) = \bigcup(E(\sigma, Q);\, Q \in S_N).$$

THEOREM 2 (F. JOHN). *If* σ *is closed and convex, then* $C(\sigma)$ *is closed and convex.*

PROOF. Since $C(\sigma)$, $D(\sigma)$ and $E(\sigma, Q)$ are all closed if σ is closed, it will suffice to prove

(L′) *If* σ *is convex,* $Q \in S_N$, $A \in E(\sigma, Q)$, $B \in C(\sigma)$ *and* $0 \leqq t \leqq 1$, *then* $(tA + (1-t)B) \in E(\sigma, Q)$.

PROOF. Applying (L) to tA and $(1-t)B$, we see that $\alpha(tA + (1-t)B)$ is in the convex hull of the set $\{t\alpha(A) + (1-t)\alpha(B)P;\ P \in S_N\}$. By hypothesis, $\alpha(A)Q$ and $a(B)PQ$ are in σ for all $P \in S_N$. Because σ is convex, $\alpha(tA + (1-t)B)Q$ is in σ. This proves (L′).

Not only does (L) directly imply (L′) but conversely and equally directly (L′) implies (L). To see this, first note that one may add arbitrary multiples of the identity map I of X onto X to A and B without affecting the validity of (L). Hence we may assume that trace $A >$ trace $B = 0$. By (L), $\alpha(A+B) = \alpha((1/2)(2A) + (1/2)(2B))$ is in the convex hull of the set $\{2\alpha(A), 2\alpha(B)P; P \in S_N\}$. Hence

$$\alpha(A + B) = 2s\alpha(A) + 2(1 - s)\beta \tag{4}$$

for $0 \leqq s \leqq 1$ and β in the convex hull of the set $\{\alpha(B)P;\ P \in S_N\}$. Summing coordinates in (4), we see that $s = 1/2$, $\alpha(A+B) = \alpha(A) + \beta$. This proves (L).

2. **Some inequalities of Amir-Moéz.** In this section we use (L) and our conservation principle to give a very direct inductive proof of the generalization of (L) due to Amir-Moéz [1].

First let us introduce some convenient notation and terminology. If (i): $i_1 \leqq i_2 \leqq \cdots \leqq i_n$ is a monotone sequence of positive integers, write (I): $I_1 < I_2 < \cdots < I_n$ for the least properly monotone majorant of (i). We call (I) the *cover* of (i) or we say that (I) *covers* (i). If also (j): $j_1 \leqq j_2 \leqq \cdots \leqq j_n$ is a monotone sequence of positive integers and N is a positive integer, we call the pair (i), (j) *N-admissible* if and only if $n \leqq N$ and $k_s = i_s + j_s - 1 \leqq N - n + s$ for $1 \leqq s \leqq n$. When the pair (i), (j) is N-admissible, we write $(i) \circ (j) = (K)$ for the cover of (k). Note that I_n, J_n, $K_n \leqq N$ which explains our term N-admissible. Using the notations of §1, we may now state the inequalities of Amir-Moéz as follows.

THEOREM 3 (AMIR-MOÉZ). *If* N *is a positive integer*; (i), (j) *is an* N*-admissible pair with covers* (I), (J), *respectively, while* $(K) = (i) \circ (j)$, *then*

$$\sum_{s=1}^{n} \tilde{\gamma}(K_s') \leqq \sum_{s=1}^{n} \tilde{\beta}(I_s') + \sum_{s=1}^{n} \tilde{\beta}(J_s'). \tag{A-M}$$

PROOF. We use induction on N, noting that for $N = 1$ or for $n = N$ the theorem is trivial and we assume 1, $n < N$.

Case 1. $I_1 > 1$ (or $J_1 > 1$). Let $\tilde{X}$ be the subspace spanned by

$e_2, \cdots, e_N$ and define $\tilde{A}$, $\tilde{B}$ on $\tilde{X}$ as in the proof of (L). Since $i_1>1$, $(i')=(i_1-1, \cdots, i_n-1)$ is a monotone sequence of positive integers whose cover is $(I')=(I_1-1, \cdots, I_n-1)$. Since $i_s>1$ for $1\leqq s\leqq n$, we easily see that (i'), (j) is an $(N-1)$-admissible pair and $J_n\leqq N-1$. Since $k_1>1$, we see that $(i)\circ(j)=(K')=(K_1-1, \cdots, K_n-1)$. By induction we have

$$\sum_{s=1}^{n} \tilde{\gamma}(K_s') \leqq \sum_{s=1}^{n} \tilde{\alpha}(I_s') + \sum_{s=1}^{n} \tilde{\beta}(J_s'). \tag{5}$$

Our conservation principle yields $\gamma(K_s)\leqq\tilde{\gamma}(K_s-1)=\tilde{\gamma}(K_s')$, $\tilde{\alpha}(I_s')=\tilde{\alpha}(I_s-1)=\alpha(I_s)$ and $\tilde{B}(J_s)\leqq\beta(J_s)$ for $1\leqq s\leqq n$. Thus (A-M) for (I), (J), (K) follows from (5).

Case 2. $i_1=j_1=1$. Define t as the greatest integer such that $I_s=s$ for $1\leqq s\leqq t$ and u as the greatest integer such that $J_s=s$ for $1\leqq s\leqq u$. By interchanging (i) and (j), as necessary, we may assume that $u\leqq t$. We see that $t<n$ by means of the following

REMARK. If we replace i_s by $i_s'\leqq i_s$ for $1\leqq s\leqq n$, then the pair (i'), (j) is N-admissible if the pair (i), (j) is N-admissible. If $K=(i')\circ(j)$, then $K_s'\leqq K_s$ for $1\leqq s\leqq n$ so that $\gamma(K_s')\geqq\gamma(K_s)$ for $1\leqq s\leqq n$. If (I'), the cover of (i') is the same as (I), then (A-M) for (I'), (J), (K') immediately yields (A-M) for (I), (J), $(K)=(i)\circ(j)$. Let us refer to this process as *irrelevant reduction of* (i).

Now suppose that $t=n$. By irrelevant reduction of (i) we may replace (i) by $i_s=1$ for $1\leqq s\leqq n$. But then $K_s=J_s$ for $1\leqq s\leqq n$ and (A-M) follows from (L). We may assume that $t<n$ and $j_{u+1}>u+1$ so that $u+N+1-J_{u+1}<N$.

By irrelevant reduction of (i) and (j) we may assume that $i_s=1$ for $1\leqq s\leqq t$ and $j_s=1$ for $1\leqq s\leqq u$. Choose an $N-1$ dimensional subspace $\tilde{X}$ which contains $g_1, \cdots, g_u$ and $f(J_{u+1}), \cdots, f(N)$. Define $\tilde{A}$, $\tilde{B}$ as in Case 1. Let $(j')=(j_1, \cdots, j_u, j_{u+1}-1, \cdots, j_n-1)$. Since $j_{u+1}-1>u=J_u$, the cover of (j') is $(J')=(J_1, \cdots, J_u, J_{u+1}-1, \cdots, J_n-1)$. A similar argument using also $i_{t+1}>t+1$ if $u=t$ shows that $(K')=(i)\circ(j')=(K_1, \cdots, K_u, K_{u+1}-1, \cdots, K_n-1)$. We may assume that $I_n<N$ since $i_s\leqq N-n+s+1-j_s$ for $1\leqq s\leqq n$ shows that $I_n\leqq N+1-j_n$ so that $I_n = N$ gives $j_n=1$, $t=u=n$ and (A-M) is a very special case of (L). By induction (A-M) holds for (I), (J'), (K'). Using our conservation principle, we easily see that (A-M) holds for (I), (J), (K). This completes the proof of Theorem 3.

3. **The Hersch-Zwahlen inequalities.** We conclude with a brief proof of the inequalities recently derived by J. Hersch [**3**] and B. P. Zwahlen [**9**].

The statement of these inequalities is somewhat complicated. Consider $(k)=\{1\leqq k_1<k_2<\cdots<k_n\leqq N\}$ and $0<p_1<p_2<\cdots<p_m=p$; $0\leqq q_1<q_2<\cdots<q_t=q$ with $p+q=n$, $t=m$ or $t=m-1$, and $q_1=0$ only if $m=1$ and $p_1=p=n$. Let $p_0=q_0=0$, $p(m+1)=q(t+1)=n$ and $k(n+s)=N+s$ for $s>n$. Define, with Zwahlen,

$$i_s = k(s+q_r)-q_r \qquad \text{for } p_r<s\leqq p_{r+1}, \qquad 0\leqq r\leqq m;$$

and

$$j_s = k(s+p_{r+1})-p_{r+1} \qquad \text{for } q_r<s\leqq q_{r+1}, \qquad 0\leqq r\leqq t.$$

THEOREM 4 (HERSCH-ZWAHLEN). *If the foregoing definitions hold, then*

$$\text{(H-Z)} \qquad \sum_{s=1}^{n}\gamma(k_s) \geqq \sum_{s=1}^{n}\alpha(i_s)+\sum_{s=1}^{n}\beta(j_s).$$

PROOF. We begin by observing that the case $q_1=0$ is merely (Λ) in its equivalent complementary form. As usual, if $n=N$, (H-Z) is an equality. Hence we may assume that $q_1>0$ and that $n<N$. We use induction on N noting that the theorem is true for $N=1$. We define v to be the greatest integer for which $k(n-s+1)=N-s+1$ for $1\leqq s\leqq v$. We may have $v=0$ which occurs if and only if $k(n)<N$. We define u to be the greatest integer for which $k_s=s$ for $1\leqq s\leqq u$. We may have $u=0$ which occurs if and only if $k(1)>1$. Observe that $v+u\leqq n\leqq N-1$. The definition of u and Zwahlen's definitions of the sequences (i) and (j) yield uniquely determined nonnegative integers w and z such that $u=w+z$, $i_s=s$ for $1\leqq s\leqq w$ and $j_s=s$ for $1\leqq s\leqq z$ while $i(w+1)>w+1$ and $j(z+1)>z+1$. We may choose a subspace $\tilde{X}$ of dimension $N-1$ containing the vectors $e_1,\cdots,e_w;f_1,\cdots,f_z$, as well as $g(N-v+1),\cdots,g(N)$. Define $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ as usual on $\tilde{X}$. If we replace N by $N-1$ and k_s by k_s-1 for $u<s$ and use Zwahlen's definitions as well as our inductive hypothesis, we obtain an inequality like (H-Z) with the corresponding replacements. But, since $\tilde{\alpha}_s=\alpha_s$ for $1\leqq s\leqq w$ and $\tilde{\alpha}(s-1)\geqq\alpha_s$ for $w<s\leqq n$; $\tilde{\beta}_s=\beta_s$ for $1\leqq s\leqq z$ and $\tilde{\beta}(s-1)\geqq\beta_s$ for $z<s\leqq n$; while $\gamma_s\geqq\tilde{\gamma}_s$ for $1\leqq s\leqq n-v$ and $\gamma_s=\tilde{\gamma}(s-1)$ for $n-v<s\leqq n$, (H-Z) follows from the corresponding inequality for $\tilde{A}$ and $\tilde{B}$.

REFERENCES

1. A. Amir-Moéz, ***Extreme properties of a hermitian transformation and singular values of the sum product of a linear transformation***, Duke Math. J. **23** (1956), 463–476.

2. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Springer-Verlag, Berlin, 1931, p. 28.

3. J. Hersch and B. P. Zwahlen, *Evaluations par défaut pour une summe de valeurs propres γ_k d'un operaton $C=A+B$ à l'aide de valeurs α_i de A et β_j de B*, C. R. Acad. Sci. Paris **254** (1962), 1559–1562.

4. A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. **12** (1962), 225–242.

5. F. John, *On symmetric matrices whose eigenvalues satisfy linear inequalities*, Proc. Amer. Math. Soc. **17** (1966), 1140–1145.

6. V. Lidskiĭ, *The proper values of the sum and product of symmetric matrices*, Dokl. Akad. Nauk SSSR **75** (1950), 769–772.

7. M. F. Smiley, *Algebra of matrices*, Allyn and Bacon, Boston, 1965, pp. 211–212.

8. H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc. **6** (1955), 106–110.

9. B. P. Zwahlen, *Über die Eigenwerte der Summe zweier selbsadjungierter Operatonen*, Comment. Math. Helv. **40** (1966), 81–116.

10. A. S. Markus, *The eigen- and singular values of the sum and product of two matrices*, Russian Math. Surveys **19** (1964), 91–120.

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