A NOTE ON SECOND ORDER DIFFERENTIAL INEQUALITIES

KEITH SCHRADER

1. **Introduction.** In the following x, y and z are variables in R, the real numbers, and f is a real valued function. We will at times assume that one or both of the following conditions hold:

(i) $f(x, y, z)$ is continuous on $S = \{(x, y, z) \in R^3 : a < x < b\}$.

(ii)′ For any y_1 and y_2 and any $a < x_1 < x_2 < b$, if the boundary value problem

$$(1) \qquad y'' = f(x, y, y'), \qquad y(x_1) = y_1, \qquad y(x_2) = y_2$$

has a solution on $[x_1, x_2]$, then it has a solution which extends throughout (a, b).

With these assumptions we are able to establish the following theorem.

THEOREM 1. *If* (i) *and* (ii)′ *hold, then each of the following are equivalent:*

(A) *For any* $a < x_1 < x_2 < b$ *and any* $\phi \in C^2[x_1, x_2]$, $\phi'' \geqq f(x, \phi, \phi')$ *is a necessary and sufficient condition for* ϕ *to be a subfunction on* $[x_1, x_2]$ *relative to solutions of* (1).

(B) *For any* $a < x_1 < x_2 < b$ *and any* $\psi \in C^2[x_1, x_2]$, $\psi'' \leqq f(x, \psi, \psi')$ *is a necessary and sufficient condition for* ψ *to be a superfunction on* $[x_1, x_2]$ *relative to solutions of* (1).

(C) *For any* $a < x_1 < x_2 < b$ *and any solutions* y *and* z *of* (1) *on* $[x_1, x_2]$ *with* $y(x_1) = z(x_1)$ *and* $y(x_2) = z(x_2)$, *it follows that* $y(x) = z(x)$ *for* $x_1 \leqq x \leqq x_2$.

In a previous paper [**3**] it was shown that if (i), (ii), and (iii) hold then (A) follows where (ii) and (iii) are the conditions:

(ii) For any y_1 and y_2 and any $a < x_1 < x_2 < b$ the boundary value problem as in (ii)′ has a solution which extends throughout (a, b), and any two solutions which agree at two distinct points are identical throughout (a, b).

(iii) Solutions of initial value problems for (1) are unique.

We see then that the (C) implies (A) part of Theorem 1 in this paper represents a strengthening of Theorems 1 and 2 in [**3**] since the existence of solutions of boundary value problems and the uniqueness of solutions to initial value problems are not explicitly hypothesized in the author's Theorem 1.

Received by the editors November 7, 1966 and, in revised form, April 27, 1967.

The author wishes to thank the referee for suggestions which contributed materially to the organization and clarity of this paper.

2. **Preliminary results.** We begin with a lemma which is critical for the ensuing proofs.

LEMMA 1. *Assume that* $f(x, y, y')$ *satisfies condition* (i) *and let* $[c, d]\subset(a, b)$, $\phi\in C^2[c, d]$ *and* $\phi''\geqq f(x, \phi, \phi')$. *Then there exists a* $\delta>0$ *such that for any* $[x_1, x_2]\subset[c, d]$, *with* $0<x_2-x_1\leqq\delta$, *the boundary value problem*

$$(2)\qquad y'' = f(x, y, y'),\qquad y(x_1) = \phi(x_1),\qquad y(x_2) = \phi(x_2)$$

has a solution $y\in C^2[x_1, x_2]$; *moreover*, $y(x)\geqq\phi(x)$ *on* $[x_1, x_2]$.

PROOF. Define $F(x, y, y')$ by

$$\begin{aligned} F(x, y, y') &= f(x, y, y') \qquad \text{for } y \geqq \phi(x), \\ &= f(x, \phi(x), y') + y - \phi(x) \qquad \text{for } y < \phi(x). \end{aligned}$$

The function $F(x,y,y')$ is continuous for $c\leqq x\leqq d$ and $|y|+|y'|<+\infty$. Choose $M=1+\max_{c\leqq x\leqq d}|\phi(x)|$, $N=1+\max_{c\leqq x\leqq d}|\phi'(x)|$ and let

$$Q = \max\{\,|F(x, y, y')| : c \leqq x \leqq d,\ |y| \leqq 2M,\ |y'| \leqq 2N\}.$$

If $\delta=\min\{(8M/Q)^{1/2}, 2N/Q\}$ then for any $[x_1, x_2]\subset[c, d]$ with $0<x_2-x_1\leqq\delta$, the boundary value problem

$$(3)\qquad y'' = F(x, y, y'),\qquad y(x_1) = \phi(x_1),\qquad y(x_2) = \phi(x_2)$$

has a solution $y\in C^2[x_1, x_2]$ by Lemma 1 of [**3**].

We now claim that $y(x)\geqq\phi(x)$ for $x_1\leqq x\leqq x_2$. If not then $\phi-y$ must have a positive maximum say at x_0 where $x_1<x_0<x_2$. But then $y'(x_0)=\phi'(x_0)$ and

$$\phi''(x_0) - y''(x_0) = \phi''(x_0) - f(x_0, \phi(x_0), \phi'(x_0)) + \phi(x_0) - y(x_0)$$

so

$$\phi''(x_0) - y''(x_0) \geqq \phi(x_0) - y(x_0) > 0$$

which contradicts having a maximum at x_0. Thus $y(x)\geqq\phi(x)$ for $x_1\leqq x\leqq x_2$ and so, by the definition of $F(x, y, y')$, we see that y is a solution to boundary value problem (2).

LEMMA 2. *If* $f(x, y, y')$ *satisfies* (i) *and* (C); *and* ϕ, δ *are as in Lemma* 1, *then* ϕ *is a subfunction, relative to solutions of* (1), *on every subinterval of* $[c, d]$ *whose length does not exceed* δ.

PROOF. This follows readily from Lemma 1.

LEMMA 3. *Assume that* $f(x, y, y')$ *satisfies condition* (i) *and let* $[c, d] \subset (a, b)$, $\psi \in C^2[c, d]$ *and* $\psi'' \leqq f(x, \psi, \psi')$. *Then there exists a* $\sigma > 0$ *such that for any* $[x_1, x_2] \subset [c, d]$, *with* $0 < x_2 - x_1 \leqq \sigma$, *the boundary value problem*

$$(4) \qquad y'' = f(x, y, y'), \qquad y(x_1) = \psi(x_1), \qquad y(x_2) = \psi(x_2)$$

has a solution $y \in C^2[x_1, x_2]$; *moreover*, $y(x) \leqq \psi(x)$ *on* $[x_1, x_2]$.

PROOF. The proof is similar to the proof of Lemma 1 so is omitted.

LEMMA 4. *If* $f(x, y, y')$ *satisfies* (i) *and* (C); *and* ψ, σ *are as in Lemma* 3, *then* ψ *is a superfunction, relative to solutions of* (1), *on every subinterval of* $[c, d]$ *whose length does not exceed* σ.

PROOF. This follows readily from Lemma 3.

3. **Proof of Theorem** 1. It is easy to see that (A) implies (C) and (B) implies (C). We will only show that (C) implies (A) since the proof that (C) implies (B) is similar. The fact that a function ϕ which is in $C^2[x_1, x_2]$ and is a subfunction on $[x_1, x_2]$ relative to solutions of (1) necessarily satisfies $\phi'' \geqq f(x, \phi, \phi')$, when (i) holds, follows easily from Lemma 3, or the proof may be found in Theorem 2 of [**4**] or Theorem 6 of [**1**]. For this reason we will consider only the sufficiency.

If $\phi \in C^2[x_1, x_2]$ and $\phi'' \geqq f(x, \phi, \phi')$ but ϕ is not a subfunction on $[x_1, x_2]$ relative to solutions of (1), then there exists an interval $[c, d] \subset [x_1, x_2]$ and a solution z of (1) with $z(c) = \phi(c)$, $z(d) = \phi(d)$ and $z(x) < \phi(x)$ for $c < x < d$.

For each positive integer n we let $P(n)$ be the proposition that that there exists an interval $[a_n, b_n] \subset [c, d]$ with $0 < b_n - a_n \leqq d - c - (n-1)\delta$ (where δ comes from Lemma 1) and a solution z_n of (1) on $[a_n, b_n]$ such that $z_n(a_n) = \phi(a_n)$, $z_n(b_n) = \phi(b_n)$ and $z_n(x) < \phi(x)$ for $a_n < x < b_n$. We will show that under our assumption that ϕ is not a subfunction on $[x_1, x_2]$ relative to solutions of (1), it follows that $P(n)$ holds for each positive integer n. This gives a contradiction since it is not possible to have $0 < d - c - (n-1)\delta$ for every positive integer n.

The fact that $P(1)$ is true follows by letting $a_1 = c$, $b_1 = d$ and $z_1 = z$. We assume that $P(k)$ is true and will show that this implies $P(k+1)$ is true. If $b_k - a_k \leqq \delta$ then we get a contradiction from Lemma 2 so we suppose $b_k - a_k > \delta$. Let y_1 be the solution to the boundary value problem

$$y'' = f(x, y, y'), \qquad y(a_k) = \phi(a_k), \qquad y(a_k + \delta) = \phi(a_k + \delta)$$

which exists on $[a_k, a_k + \delta]$ by Lemma 1. By condition (ii)$'$ this boundary value problem has a solution which exists on (a, b) and by

(C) this solution on (a, b) must agree with y_1 on $[a_k, a_k+\delta]$ so we see that y_1 is extendable as a solution of (1) to all of (a, b). If $P(k+1)$ is not true then one may show that $y_1(x) \geqq \phi(x)$ for $a_k \leqq x \leqq b_k$. To see this first note that $y_1(x) \geqq \phi(x)$ on $[a_k, a_k+\delta]$ by Lemma 1 and that $y_1(b_k) > \phi(b_k)$ by (C). Now we must have $y_1(x) \geqq \phi(x)$ on $[a_k+\delta, b_k]$ or we violate the assumption that $P(k+1)$ is not true. If $b_k - a_k - \delta \leqq \delta$ we observe that the solution y_2 to the boundary value problem

$$y'' = f(x, y, y'), \qquad y(a_k + \delta) = \phi(a_k + \delta), \qquad y(b_k) = \phi(b_k)$$

which exists on $[a_k+\delta, b_k]$ by Lemma 1 is extendable to all of (a, b) just as y_1 was. We notice that $y_2(x) \geqq \phi(x)$ for $a_k+\delta \leqq x \leqq b_k$ by Lemma 1 and that $y_2(x) \leqq y_1(x)$ for $a_k+\delta \leqq x \leqq b_k$ by (C). We now have $\phi(x) \leqq y_2(x) \leqq y_1(x)$ for $a_k+\delta \leqq x \leqq b_k$, $y_1(a_k+\delta) = y_2(a_k+\delta) = \phi(a_k+\delta)$ and $y_1'(a_k+\delta) = \phi'(a_k+\delta)$. Using these properties and elementary calculus it is not hard to show that $y_2'(a_k+\delta) = y_1'(a_k+\delta)$ and then since y_1 and y_2 are both solutions of (1), that $y_2''(a_k+\delta) = y_1''(a_k+\delta)$. But now condition (C) is violated since u defined by

$$\begin{aligned} u(x) &= y_1(x) \qquad \text{for } a_k \leqq x \leqq a_k + \delta, \\ &= y_2(x) \qquad \text{for } a_k + \delta < x \leqq b_k \end{aligned}$$

is a solution to the same boundary value problem that z_k is, but u and z_k are not identical. We conclude that $b_k - a_k - \delta > \delta$ and let y_2 be the solution to the boundary value problem

$$y'' = f(x, y, y'), \qquad y(a_k + \delta) = \phi(a_k + \delta), \qquad y(a_k + 2\delta) = \phi(a_k + 2\delta)$$

which exists on $[a_k+\delta, a_k+2\delta]$ by Lemma 1 and is extendable to all of (a, b) just as y_1 was. Note that $y_2(x) \geqq \phi(x)$ for $a_k+\delta \leqq x \leqq b_k$ for otherwise the function v defined by

$$\begin{aligned} v(x) &= y_1(x) \qquad \text{for } a_k \leqq x \leqq a_k + \delta, \\ &= y_2(x) \qquad \text{for } a_k + \delta < x \leqq b_k \end{aligned}$$

is a solution of (1) that must satisfy $v(b_k) > \phi(b_k)$, or (C) is violated, and hence $v(x) \geqq \phi(x)$ for $a_k+\delta \leqq x \leqq b_k$ or else the assumption that $P(k+1)$ is false is violated. Continuing in this way we construct $y_3, y_4, \cdots$ until we have worked our way across the interval $[a_k, b_k]$ and obtained a contradiction. Thus $P(k+1)$ is true as claimed but this gives a contradiction also, so the proof is complete.

4. Further results.

THEOREM 2. *If condition* (ii)′ *is omitted in the hypotheses of Theorem* 1, *then the conclusion remains valid provided the condition* $x_2 - x_1 \leqq \delta$ *is added in each of* (A), (B) *and* (C).

PROOF. The proof is the same as that of Theorem 1 except for the sufficiency of $\phi'' \geqq f(x, \phi, \phi')$ for ϕ to be a subfunction, relative to solutions of (1) on $[x_1, x_2]$ with $x_2 - x_1 \leqq \delta$ when (C) is assumed to hold. This now follows from Lemma 2.

We now consider a condition which is weaker than (ii)$'$:

(ii)$''$ for any y_1 and y_2 and any $a < x_1 < x_2 < b$, if the boundary value problem as in (ii)$'$ has a solution on $[x_1, x_2]$, then it has a solution y which extends to the right until $x = b$ or to $x_0 < b$ where $\limsup_{x \to x_0 -} y(x) = +\infty$ or $\liminf_{x \to x_0 -} y(x) = -\infty$ and similarly to the left.

THEOREM 3. *If* (i) *and* (ii)$''$ *hold then the statements* (A), (B) *and* (C) *of Theorem* 1 *are equivalent.*

PROOF. The proof is nearly identical to the proof of Theorem 1, so it is omitted.

COROLLARY. *If f satisfies* (i) *but does not depend on z then the statements* (A), (B) *and* (C) *of Theorem* 1 *are equivalent.*

PROOF. If f satisfies (i) but does not depend on z then one can show, using Theorem 3.1 [2, p. 12], that (ii)$''$ holds.

DEFINITION. For any $a < x_1 < x_2 < b$, a function $\phi \in C^2[x_1, x_2]$ will be called a lower solution of (1) on $[x_1, x_2]$ in case $\phi'' \geqq f(x, \phi, \phi')$ for $x \in [x_1, x_2]$. Similarly, $\psi \in C^2[x_1, x_2]$ will be called an upper solution of (1) on $[x_1, x_2]$ in case $\psi'' \leqq f(x, \psi, \psi')$ for $x \in [x_1, x_2]$.

THEOREM 4. *If* (i) *and either* (ii)$'$ *or* (ii)$''$ *hold then the following are equivalent*:

(A)$'$ *For any $a < x_1 < x_2 < b$ and any $\phi \in C^2[x_1, x_2]$, $\phi'' \geqq f(x, \phi, \phi')$ is a necessary and sufficient condition for ϕ to be a subfunction on $[x_1, x_2]$ relative to upper solutions of* (1).

(B)$'$ *For any $a < x_1 < x_2 < b$ and any $\psi \in C^2[x_1, x_2]$, $\psi'' \leqq f(x, \psi, \psi')$ is a necessary and sufficient condition for ψ to be a superfunction on $[x_1, x_2]$ relative to lower solutions of* (1).

(C) *For any $a < x_1 < x_2 < b$ and any solutions y and z of* (1) *on $[x_1, x_2]$ with $y(x_1) = z(x_1)$ and $y(x_2) = z(x_2)$, it follows that $y(x) = z(x)$ for $x_1 \leqq x \leqq x_2$.*

PROOF. The only part of the proof which differs from the proof of Theorem 1 is the sufficiency of $\phi'' \geqq f(x, \phi, \phi')$ for ϕ to be a subfunction on $[x_1, x_2]$ relative to upper solutions of (1) when (C) is assumed to hold. We will show this assuming (ii)$'$ holds since the proof when (ii)$''$ holds is similar.

If $\phi \in C^2[x_1, x_2]$ and $\phi'' \geqq f(x, \phi, \phi')$ but ϕ is not a subfunction on $[x_1, x_2]$ relative to upper solutions of (1), then there exists an interval

$[c, d] \subset [x_1, x_2]$ and an upper solution ψ of (1) with $\psi(c) = \phi(c)$, $\psi(d) = \phi(d)$ and $\psi(x) < \phi(x)$ for $c < x < d$.

For each positive integer n we let $P(n)$ be the proposition that there exists an interval $[a_n, b_n] \subset [c, d]$ with $0 < b_n - a_n \leqq d - c - (n-1)\eta$ (where $\eta = \min\{\delta, \sigma\}$ and δ, σ come from Lemmas 1 and 3) and an upper solution z_n of (1) on $[a_n, b_n]$ such that $z_n(a_n) = \phi(a_n)$, $z_n(b_n) = \phi(b_n)$ and $z_n(x) < \phi(x)$ for $a_n < x < b_n$. We will show that under our assumption that ϕ is not a subfunction on $[x_1, x_2]$ relative to upper solutions of (1), it follows that $P(n)$ holds for each positive integer n. This gives a contradiction since it is not possible to have $0 < d - c - (n-1)\eta$ for every positive integer n.

The fact that $P(1)$ is true follows by letting $a_1 = c$, $b_1 = d$ and $z_1 = \psi$. We assume that $P(k)$ is true and will show that this implies $P(k+1)$ is true. If $b_k - a_k \leqq \eta$ then we get a contradiction from Lemmas 1, 3 and property (C) so we suppose $b_k - a_k > \eta$. From this point on, the proof proceeds as in Theorem 1 except that η replaces δ and in several places where (C) is cited as the reason for something being true, one needs to use either (A) or (B) of Theorem 1 which are each equivalent to (C), under our hypotheses, by Theorem 1 (Theorem 3 in the case where (ii)″ is assumed to hold).

These results raise the question of whether the conclusion of Theorem 1 is valid if condition (ii)′ is omitted from the hypotheses entirely. The author does not know the answer to this conjecture; however, it is not hard to see that the function $f(x, y, z)$ in any counterexample must be nonlinear in y or z, cannot be strictly increasing in y for each fixed x and z, must depend on z and cannot satisfy a so called "Nagumo condition" as described, for example, in Lemma 5.1 [2, p. 428].

References

1. L. Fountain and L. Jackson, *A generalized solution of the boundary value problem for* $y'' = f(x, y, y')$, Pacific J. Math. **12** (1962), 1251–1272.

2. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

3. L. Jackson and K. Schrader, *On second order differential inequalities*, Proc. Amer. Math. Soc. **17** (1966), 1023–1027.

4. M. Peixoto, *Generalized convex functions and second order differential inequalities*, Bull. Amer. Math. Soc. **55** (1949), 563–572.

University of Missouri