COMPACTIFICATIONS OF n-SPACE BY AN ARC

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1. **Introduction.** We ask how many different spaces X we obtain by compactifying n-space R^n by an arc A. (We are not asking how many distinct pairs (X, A) exist.) For n large, most certainly uncountably many distinct X ought to exist. However, if we require that A be fitted nicely to R^n, the question becomes more interesting and perhaps harder. (On the other hand, if X is required to be a topological manifold, then it must be a sphere in every dimension where the topological Poincaré conjecture is true.) We solve one aspect of this problem in the following form.

THEOREM. *For each $n \geqq 4$, there exist uncountably many topologically distinct cohomology (or generalized) n-manifolds over integers Z which are obtained from R^n by compactifying by an arc.*

By [6, Chapter IX], X is necessarily n-sphere for $n=1$ and 2, the question is left open only for the case $n=3$. The proof makes a strong use of the method of [1] and constructions of [1] and [2]. As a corollary to [1], we also obtain uncountably many generalized n-cells, $n \geqq 4$, which are AR but whose boundaries are not 1-LC.

2. **Contractible manifolds with boundary.** Let Λ denote the collection of all sequences of the form

$$\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \cdots, \{a_1, a_2, \cdots, a_k\}, \{a_1, \cdots\},$$

where $a_1 < a_2 < a_3 < \cdots$ are all positive integers. The ith term of each $\lambda \in \Lambda$ will be denoted by $\lambda(i)$. We emphasize the following three properties of Λ:

(1) For any integer i and $\lambda \in \Lambda$ there exist infinitely many integers $i = i_1 < i_2 < \cdots$ such that $\lambda(i_q) = \lambda(i)$.

(2) If $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$, then there is a term in one of λ and λ' which does not appear as terms in the other.

(3) The set Λ is uncountable.

Let n be a fixed integer greater than 3. Let $M_1, M_2, \cdots$ be compact contractible n-manifolds such that $\pi_1(\text{Bd } M_i)$ are pairwise nonisomorphic, each $\pi_1(\text{Bd } M_i) \neq 1$ and no $\pi_1(\text{Bd } M_i)$ is isomorphic to the free product of two nontrivial groups. The existence of such manifolds has been shown in [1], [2]. Further we assume that $M_i \times [0, 1]$ is an

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$(n+1)$-cell. (See [1], [2].) For each $\lambda \in \Lambda$, consider the contractible n-manifolds

$$M_\lambda = M_{\lambda(1)} \# M_{\lambda(2)} \# \cdots,$$

the infinite connected sum as defined in [1].

3. **Compactifications of R^n by an arc.** For each $\lambda \in \Lambda$, construct the space X_λ which is obtained as follows. Let N_λ be the one-point compactification of M_λ by a point p_λ. $X_\lambda \subset N_\lambda \times [0, 1]$ is defined to be the set

$$[(\mathrm{Bd}\, M_\lambda) \cup \{p_\lambda\}] \times [0, 1] = N_\lambda \times \{0, 1\}.$$

PROPOSITION 1. *X_λ is a spherelike cohomology manifold over Z.*

PROOF. By [6], Bd $M_\lambda \cup \{p_\lambda\}$ is a spherelike $(n-1)$-cm. (cm stands for cohomology manifold over any principal ideal domain. Henceforth, the mention of the coefficient domains will be omitted.) Now if we take two copies of M_λ and attach them along the boundaries, we obtain a space homeomorphic to R^n. (Recall $M_i \times [0, 1]$ is an $(n+1)$-cell.) We write this fact as $2M_\lambda = R^n$. Compactify $2M_\lambda$ by the point p_λ. Then Bd $M_\lambda \cup \{p_\lambda\} \subset$ n-sphere which is the one-point compactification of $2M_\lambda = R^n$. The closure of either complementary domain is homeomorphic to N_λ. By [7, Theorem 9.1, p. 312] or [5, Theorem 2, p. 12] and [3, p. 434], N_λ is a generalized n-cell in the sense of [7] and [3]. Consequently (see [3]), $N_\lambda \times [0, 1]$ is a generalized $(n+1)$-cell and $X_\lambda = \mathrm{Bd}(N_\lambda \times [0, 1])$ is a spherelike n-cm.

Now let A_λ be the arc in X_λ corresponding to $p_\lambda \times [0, 1]$.

PROPOSITION 2. *$X_\lambda - A_\lambda$ is homeomorphic to R^n.*

PROOF. Since $M_i \times [0, 1]$ is an $(n+1)$-cell, $M_\lambda \times [0, 1]$ is an infinite connected sum of $(n+1)$-cells and therefore is homeomorphic to $R^n \times [0, 1)$. On the other hand $X_\lambda - A_\lambda$ is homeomorphic to Bd $(N_\lambda \times [0, 1] - A_\lambda)$. Hence it is homeomorphic to Bd $(R^n \times [0, 1))$ $= R^n \times 0$.

In order to prove the theorem in the Introduction, it suffices to show that X_λ and X'_λ are distinct if $\lambda \neq \lambda'$.

4. **Proof that X_λ and X'_λ are not homeomorphic.**

PROPOSITION 3. *$A_\lambda \subset X_\lambda$ is precisely the set of points of X_λ at which X_λ is not locally euclidean.*

PROOF. We observe that N_λ is not locally 1-connected at p_λ in the sense of homotopy. Hence X_λ is not locally 1-connected at any interior point of A_λ. Hence X_λ is not locally euclidean at any point of A_λ.

In view of Proposition 3, it suffices to show there is no homeomorphism h of (X'_λ, A'_λ) onto (X_λ, A_λ).

PROPOSITION 4. *If $\lambda \neq \lambda'$, there is no homeomorphism h of (X'_λ, A'_λ) onto (X_λ, A_λ).*

PROOF. We use a simpler version of an argument in [1]. Because of a difference in situation (for instance, we do not have here group systems in the sense of [1]) we will reproduce a certain argument of [1] to show that such an argument in fact goes through in our present case. However, the reader's familiarity with [1] is assumed throughout. Suppose there exists such a homeomorphism h.

Let

$$U_i = \text{Bd } N_\lambda - (M_{\lambda(1)} \# M_{\lambda(2)} \# \cdots \# M_{\lambda(i-1)}).$$

Let G_i be an abstract group isomorphic to $\pi_1(\text{Bd } M_{\lambda(i)})$. With suitably chosen base points, we may write

$$\pi_1(U_i') = G_i * G_{i+1} * \cdots,$$

the infinite free product, where $U_i' = U_i - P_\lambda$. In order to avoid confusion, $G_j \subset \pi_1(U_i')$ will be denoted by G_j^i. Though it is impossible that all $\pi_1(U_i')$ have a common base point, it is nevertheless possible to have a homomorphism of $\pi_1(U'_{i+1}, x_{i+1}) \to \pi_1(U_i', x_i)$ by $\pi_1(U'_{i+1} X_{i+1}) \to \pi_1(U_i', x_{i+1}) \to \pi_1(U_i', x_i)$ where the first homomorphism is induced by the inclusion and the second one by a path in U_i' connecting x_i and x_{i+1}. See [1]. This way we define homomorphisms

$$\pi_1(U'_{i+1}) \to \pi_1(U_i'),$$

or more generally,

$$\pi_1(U'_{i+k}) \to \pi_1(U_i').$$

Under the latter homomorphism, G_j^{i+k} maps isomorphically onto G_j^i. See [1, p. 38].

Let $a = (p_\lambda, 1/2) \in A_\lambda$ and $P_i = U_i \times (1/2 - 1/2^i, 1/2 + 1/2^i)$. It is possible to identify

$$\pi_1(P_i') = G_i^i * G_{i+1}^i * \cdots$$

and define homomorphisms

$$\pi_1(P'_{i+k}) \to \pi_1(P_i'),$$

where $P_j' = P_j - A_\lambda$ just as for U_i's. Let $b \in A_\lambda'$ be such that $h(b) = a$ and $Q_1Q_2 \cdots$ be a sequence of neighborhoods of b having the same relation to b as $P_1, P_2, \cdots$ do to a.

Let $Q_j' = h(Q_j) - A_\lambda$. Then $\pi_1(Q_j') = F_j * F_{j+1} * \cdots$, where $F_k \simeq \pi_1(\text{Bd } M'_{\lambda(k)})$. Find integers j, k, s, t such that $P_i' \supset Q_s' \supset P_j' \supset Q_t' \supset P_k'$. As in [1], it is possible to find homomorphisms

$$\begin{aligned} \pi_1(P_k') &= G_k^k * G_{k+1}^k * \cdots \\ &\quad \downarrow f \\ \pi_1(Q_t') &= F_t^t * F_{t+1}^t * \cdots \\ &\quad \downarrow g \\ \pi_1(P_j') &= G_i^i * G_{i+1}^i * \cdots \\ &\quad \downarrow f' \\ \pi_1(Q_s') &= F_s^s * F_{s+1}^s * \cdots \end{aligned}$$

such that gf is the monomorphism we defined above up to a conjugacy automorphism of $\pi_1(P_j')$ and $f'g$ is the monomorphism similarly definable for $Q's$ up to a conjugacy automorphism of $\pi_1(Q's)$.

Without loss of generality, we may suppose that G_k^k is not isomorphic to any F_m. The bar above homomorphisms will denote appropriate restrictions. Since G_k^k is not a free product of two nontrivial groups, by Kurosh's subgroup theorem [3], $f(G_k^k)$ is a conjugate of some subgroup of some F_m^t. Then since $gf(G_k^k)$ is a conjugate of G_k^i, $g(F_m^t)$ is a conjugate of G_k^i we have

$$G_k^k \xrightarrow{\bar{f}} \text{conjugate of } F_m^t \xrightarrow{\bar{g}} \text{conjugate of } G_k^i.$$

Since $\bar{f}$, $\bar{g}$ are monomorphisms and $\bar{g}\bar{f}$ is an isomorphism, $\bar{f}$ is an isomorphism. Thus $G_k^k \simeq F_m$, a contradiction.

5. **Some generalized n-cells.** Consider $N_\lambda \times [0, 1]$. If we shrink A_λ to a point, we obtain an $(n+1)$-cell. In this $(n+1)$-cell, the subset corresponding to $N_\lambda \times 0$ sits as a strong deformation retract. Hence N_λ is an AR (absolute retract). This leads to

COROLLARY. *For each $n \geqq 4$, there exist uncountably many generalized n-cells N_λ which are AR but whose boundaries are not 1-LC.*

REMARK. N_λ is never a cartesian factor of a cell. If $N_\lambda \times C$ is a cell, Bd $N_\lambda \times$ Int C would be an open subset of a sphere and Bd N_λ would be locally contractible.

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