ON SUSPENDING HOMOTOPY SPHERES

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A *homotopy n-sphere* is a compact topological n-manifold having the homotopy type of S^n and a *fake n-cell* a compact, contractible, topological n-manifold whose boundary is homeomorphic to S^{n-1}. The object of this paper is to establish the following two propositions as regards such classes of manifolds.

THEOREM 1. *The suspension of a homotopy* 4-*sphere is homeomorphic to* S^5.

THEOREM 2. *The suspension of a fake* 4-*cell is homeomorphic to* I^5.

It should be noted that M. Hirsch [4] has proved both theorems for the case in which the manifolds are smooth by the methods of differential topology. The case for piecewise linear (combinatorial) manifolds follows immediately as a result of their admitting compatible smooth structures.

We prove Theorem 2 first as follows: Let F^4 be a fake 4-cell and h: Bd $F^4 \times [0, 1) \to F^4$ a collar for Bd F^4 in F^4. Put $X = \mathrm{Cl}(F^4 - h[0, 1/2])$. Note that X is homeomorphic to F^4. Let $N^5 = F^4 \times [-2, 2]$. We claim that Int N^5 is a contractible open 5-manifold which is 1-connected at infinity, admits a piecewise linear triangulation and thus by Stallings [6] is homeomorphic to E^5. Obviously the interior of N^5 is contractible and since by Van Kampen's theorem Bd N^5 is simply connected (it is in fact a homotopy 4-sphere) Int N^5 is 1-connected at infinity. It is easily seen that the double $2N^5$ of N^5 is a homotopy 5-sphere and thus by Poincaré's conjecture for topological n-manifolds ($n \geqq 5$), which was proved by E. H. Connell and can be found in M. H. A. Newman [5], a 5-sphere. Hence Int N^5 is homeomorphic to an open subset of S^5. Consequently, it admits a piecewise-linear triangulation and is homeomorphic to E^5. Clearly $X \times (-1)$ and $X \times 1$ are cellular in Int N^5, for they have arbitrarily small neighborhoods homeomorphic to Int N^5. It follows (Theorem 1 of [1]) that there is a mapping F: $N^5 \to N^5$ such that $F|$ Bd $N^5 = \mathrm{id}$ and the only nondegenerate inverse sets of F are $X \times (-1)$ and $X \times 1$. Let g: Bd $X \times [-1, 1] \to S(\mathrm{Bd}\, X)$ be the natural mapping. Fg^{-1} is a homeomorphism of $S(\mathrm{Bd}\, X)$ onto $F(\mathrm{Bd}\, X \times [-1, 1])$. Hence $F(\mathrm{Bd}\, X \times [-1, 1])$ is a 4-sphere. Clearly $F(\mathrm{Bd}\, X \times [-1, 1])$ is locally flat in Int N^5 except

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possibly at $F(\mathrm{Bd}\, X\times -1)$ and $F(\mathrm{Bd}\, X\times 1)$. Hence by Chernavskii [3] (or by results submitted to the *Annals of Mathematics* by R. C. Kirby) locally flat. By the Generalized Schoenflies Theorem $F(X\times[-1, 1])$ is a cell. If $G: X\times[-1, 1]\to S(X)$ is the natural map (which extends g), then $F\,G^{-1}$ is a homeomorphism of $S(X)$ onto the 5-cell $F(X\times[-1, 1])$. Since X is homeomorphic to F^4, $S(F^4)$ is a 5-cell.

Theorem 1 now follows by removing from a homotopy 4-sphere M^4 the interior of a 4-simplex (4-cell with locally flat boundary) σ^4 to obtain a fake 4-cell F^4. By Theorem 1, $S(F^4)$ is a cell. Hence $S(M^4) = S(F^4)\cup S(\sigma^4)$ where $S(F^4)\cap S(\sigma^4) = S(\mathrm{Bd}\,\sigma^4)$ and is thus a 5-sphere.

Theorem 2 also follows from Theorem 1 by reversing the argument above and using the following theorem of J. C. Cantrell [2].

THEOREM 3. *If the $(n-1)$-sphere $(n>3)$ S in S^n is locally collared at a point p in one side and locally collared in a deleted neighborhood of p on the other side, then S is locally flat at p.*

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