MORE CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Let X be a real inner product space of dimension at least three and let M be a 2-dimensional subspace of X. For a vector u in X but not in M, let v be the vector in M closest to u. It is easily seen that (i) if $v=0$, then all of the vectors of norm 1 in M are equidistant from u and (ii) if $v\neq 0$ and $w=|v|^{-1}v$, then of all the vectors of norm 1 in M w is the closest to u. The purpose of this paper is to show that each of these properties characterize those normed linear spaces which are inner product spaces. (For a survey of such results, see [**3**, pp. 115–121].)

Throughout, we let E denote real Euclidean 3-space. Our proofs are based on the following two characterizations of ellipsoids in E. Theorem A is due to G. Birkhoff [**1**]. Theorem B is due to Marchaud [**4**] and generalizes a result due to Blaschke [**2**, pp. 157–159].

(A) *Let K be a compact convex body in E with bounding surface S. Suppose there exists a point* 0 *interior to K satisfying: for any line m through* 0 *and point P in $m\cap S$, if M is a plane through* 0 *so that its translate through P supports K, then for skew cylindrical coordinates (r, θ, z) with m the line $r=0$ and M the plane $z=0$, the equation of S is of the form $r=f(z)\cdot g(\theta)$. Then K is an ellipsoid.*

(B) *Let K be a compact convex body in E with bounding surface S satisfying: for every direction d in E, there exists a corresponding plane M_d such that the cylinder in the direction d generated by the plane curve $S\cap M_d$ circumscribes K. Then K is an ellipsoid.*

THEOREM. *Let X be a real normed linear space of dimension at least three. If X satisfies either condition* (1) *or* (2) *below, then X is an inner product space.*

(1) *For every 2-dimensional subspace M of X and vector u not in M for which $|u| =\min\{|u-w| : w \text{ in } M\}$, we have $|u-w| =|u-w'|$ for all w and w' in M with $|w| =|w'| =1$.*

(2) *For every 2-dimensional subspace M of X and vector u not in M for which there exists a vector v in M, $v\neq 0$, satisfying $|u-v| =\min\{|u-w| : w \text{ in } M\}$, we have*

$$|u-|v|^{-1}v| = \min\{|u-w| : w \text{ in } M,\ |w| =1\}.$$

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PROOF. It suffices to show that for any 3-dimensional subspace Y of X and any one-to-one linear mapping of Y onto E, the image K of the unit ball in Y is an ellipsoid (cf. [3]). For simplicity, we shall assume that Y is E. The first of the above conclusions follows from Theorem A and the second from Theorem B. The arguments are similar and we furnish only the latter.

Given a direction d, let m be the line through 0 (the origin) in the direction d. Let N be any plane containing m, let n be a line in N parallel to m which supports $K\cap N$, and let x be any point of $K\cap n$. Let N' be a plane parallel to N which supports K and let y be any point of $K\cap N'$. By the symmetry of K, the plane N'' parallel to N and containing $-y$ will also support K. We wish to show that the 2-dimensional subspace M of E spanned by x and y has the desired property of M_d in (B). Thus, for $S=$boundary K, we need to show that for any point z in $S\cap M$, the line p through z parallel to m supports K. Suppose $z=ax+by$ and $|z|=1$. If $a=0$ or $b=0$, then $z=\pm x$ or $\pm y$ and it is clear that p has the desired property. Assume $a, b\neq 0$; by the symmetry of K, we need only consider $a>0$. Let $u=(1/2)x-(b/2a)y$ and let

$$K_z = \{w: w \text{ in } E, \ |w-u| \geqq (1/2a)\}.$$

If T is the mapping in E defined by $T(w)=(1/2a)w+u$, then $T(0)=u$, $T(K)=K_z$ and $T(z)=x$. Since T is a magnification followed by a translation to show that p supports K at z, it suffices to show that n supports K_z at x.

We note that the ball centered at u of radius $|b|/2a$ is supported by N at $v=(1/2)x$. Thus, v is in K_z and by condition (2), $|u-x| = \min\{|u-w| : w \text{ in } N, |w|=1\}$. Suppose n does not support K_z. Then there must be a point w_0 common to n and $i(K_z)$, the interior of K_z. It follows that all points of the segment $[w_0, v]$, except possibly v, belong to $i(K_z)$. But the segment $[w_0, v]$ must contain a vector w_1 of norm 1. We then have $|u-w_1|<|u-x|$, a contradiction. Thus, n supports K_z at x and our conclusion follows.

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