CURVATURE OF NONLINEAR CONNECTIONS

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1. **Introduction.** The curvature of a homogeneous (nonlinear) connection on a vector bundle (and in particular, on the tangent bundle) has been defined by several authors in terms of local reference frames. (See [1] and the references listed there.)

In this note, an intrinsic definition is given for the curvature of a general nonlinear connection on a smooth (C^∞) vector bundle (modeled on a Banach space [3]).

Let $p\colon E\to X$ be a smooth vector bundle over a smooth manifold X. A (general nonlinear) *smooth connection* on it [5] is a smooth splitting of the (direct) exact sequence

$$0\to VE \xrightarrow{J} TE \xrightarrow{p'} p^{-1}TX\to 0 \tag{1}$$

of vector bundles over the manifold E (where $p^{-1}TX$ denotes the pullback, p' is defined by factoring the tangent map $p_*\colon TE\to TX$ through $p^{-1}TX$, and VE is the kernel of p' (or of p_*) with J its inclusion).

The splitting is given by a smooth morphism $V\colon TE\to VE$ such that $VJ=I$, or equivalently by a smooth morphism $W\colon p^{-1}TX\to TE$ such that $p'W=I$. V and W (the left and right splitting maps, respectively) are related by $JV+Wp'=I$. In other words, $V=JV$ and $H=Wp'$ are the projection maps of a direct sum decomposition $TE=HE\oplus VE$ (where $HE=$kernel $W=$image V). Of course HE, the horizontal bundle, is isomorphic to $p^{-1}TX$.

On the other hand, the vertical bundle, VE, is canonically isomorphic to $p^{-1}E$. Hence there is a canonical map $r\colon VE\to E$ over p (isomorphic on the fibres). $D=rV\colon TE\to E$ is the *connection map*. It is a smooth morphism of the tangent bundle structure on TE, and is also fibre-preserving for the fibres of the other fibre bundle structure, $p_*\colon TE\to TX$, on TE. If D is linear on the p_* fibres, one has a *linear connection*; if D is merely 1-homogeneous, then a *homogeneous connection* (also called a *nonlinear connection* by most authors).

REMARK. If $E_0\subset E$ is an open set, a smooth splitting of (1) restricted over E_0 is a *smooth connection on* E_0. This added generality is needed for strictly nonlinear connections. Namely, a smooth homogeneous connection is always assumed to be on $E_0=E-0$ (because one on E is necessarily linear).

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The left splitting map V is a vertical bundle valued 1-form on the manifold E; hence V is the *connection form*. The *curvature form* is its exterior derivative, dV, taken with respect to the linear Berwald connection on the vertical bundle, which will be shown to exist next.

2. **The Berwald connection.** Let U be the domain of a chart on X (and identify it with its homeomorphic image in the model space $\boldsymbol{B}$ of X); let $TX|U \approx U\times\boldsymbol{B}$ be the tangent chart. Suppose $E|U \approx U\times\boldsymbol{E}$ is a chart on E; by taking tangent maps one gets a chart $TE|(E|U)\approx U\times\boldsymbol{E}\times\boldsymbol{B}\times\boldsymbol{E}$. The sequence (1) over $E|U$ becomes the sequence

$$0\rightarrow U\times\boldsymbol{E}\times 0\times\boldsymbol{E}\rightarrow U\times\boldsymbol{E}\times\boldsymbol{B}\times\boldsymbol{E}\rightarrow U\times\boldsymbol{E}\times\boldsymbol{B}\rightarrow 0$$

of bundles over $U\times\boldsymbol{E}$, with $p'(x,a,\lambda,b)=(x,a,\lambda)$. The map $r\colon VE\rightarrow E$ is locally $r(x,a,0,b)=(x,b)$ and $V(x,a,\lambda,b)=(x,a,0,b+\omega(x,a)\lambda)$, whence $D(x,a,\lambda,b)=(x,b+\omega(x,a)\lambda)$.

Here the smooth map $\omega\colon U\times\boldsymbol{E}\rightarrow L(\boldsymbol{B},\boldsymbol{E})$ is the *local component* of the connection for this chart [**5**]. The connection is linear or homogeneous iff each ω is, in its second variable, a. In the linear case $\Gamma(x)(a,\lambda)=\omega(x,a)\lambda$ defines a smooth map $\Gamma\colon U\rightarrow L^2(\boldsymbol{E},\boldsymbol{B};\boldsymbol{E})$, the *local Christoffel component.*

Note if the connection is on $E_0\subset E$, ω is defined on $U\times\boldsymbol{E}_U\subset U\times\boldsymbol{E}$; e.g. $\boldsymbol{E}_U=\boldsymbol{E}-0$ in the homogeneous case. Let ∂_i denote the ith partial (Fréchet) derivative (written as D_i in [**3**, Chapter 1]).

PROPOSITION. *For a smooth connection on $E_0\subset E$, the maps $\Omega\colon(U\times\boldsymbol{E}_U)\times\boldsymbol{E}\rightarrow L(\boldsymbol{B}\times\boldsymbol{E},\boldsymbol{E})$ defined by $\Omega((x,a),b)(\mu,c)=\partial_2\omega(x,a)(b)\mu$ are the local components of a linear connection on $VE|E_0\rightarrow E_0$ (the Berwald connection).*

PROOF. If $P(x,a)=(fx,A(x)a)$ is a change of charts $U\times\boldsymbol{E}\approx U\times\boldsymbol{E}$ on E, the corresponding change of charts P_* on TE is given by

$$P_*(x,a,\lambda,b)=(fx,A(x)a,f'(x)\lambda,A'(x)(\lambda)a+A(x)b)$$

with primes denoting (Fréchet) derivatives. The local equation for D shows that the old and new local components ω and $\bar{\omega}$ are related by the classical equation

$$A'(x)(\lambda)a+\bar{\omega}(fx,A(x)a)f'(x)\lambda = A(x)\omega(x,a)\lambda.$$

Differentiating partially with respect to a in direction b and setting $\lambda=\mu$ produces

$$A'(x)(\mu)b+\partial_2\bar{\omega}(fx,A(x)a)(A(x)b)f'(x)\mu = A(x)\partial_2\omega(x,a)(b)\mu. \tag{2}$$

Since locally $T(VE)\approx(U\times\boldsymbol{E}\times 0\times\boldsymbol{E})\times\boldsymbol{B}\times\boldsymbol{E}\times 0\times\boldsymbol{E}$, a connection map for a connection on $VE|E_0\to E_0$ is locally $\boldsymbol{D}(x, a, 0, b; \mu, c, 0, d) = (x, a; 0, d+\Omega((x, a), b)(\mu, c))$, where $\Omega: (U\times\boldsymbol{E}_U)\times\boldsymbol{E}\to L(\boldsymbol{B}\times\boldsymbol{E}, \boldsymbol{E})$ is smooth. The induced change of charts P_{**} on $T(VE)$ is $P_{**}(x, a, 0, b; \mu, c, 0, d) = (fx, A(x)a, 0, A(x)b; f'(x)\mu, A'(x)(\mu)a + A(x)c, 0, A'(x)(\mu)b+A(x)d)$. Hence the old and new Ω and $\bar{\Omega}$ are related by

$$A'(x)(\mu)b + \bar{\Omega}((fx, A(x)a), A(x)b)(f'(x)\mu, A'(x)(\mu)a + A(x)c) = A(x)\Omega((x, a), b)(\mu, c).$$

For Ω defined by $\Omega((x, a), b)(\mu, c) = \partial_2\omega(x, a)(b)\mu$, (2) says this equation is satisfied. Note Ω is linear in its second variable, b. q.e.d.

REMARK 1. A conceptual existence proof goes as follows. Let $V_1: T(TE)\to V(TE)$ be the connection form of the induced connection on $p_*: TE\to TX$, [5, Theorem 1], where $V(TE) = \text{kernel } p_*$. Since $VE\subset TE$, V_1 restricts to a map $V_1: T(VE)\to V(TE)|VE$. Now this latter bundle is canonically isomorphic to the bundle $v_1^{-1}(v^{-1}VE)$, where $v: VE\to E$ and $v_1: v^{-1}VE\to VE$. But $v^{-1}VE\approx V(VE) = \text{kernel } v$. Hence there is a canonical epimorphism $g: V(TE)|VE\to V(VE)$, and gV_1 is the connection form of a linear connection on $v: VE\to E$.

REMARK 2. The original Berwald connection occurs in the case $E = TX$ and the connection is the canonical (homogeneous) connection of a smooth Finsler structure on X. (See the references in [1].)

3. **The curvature form.** For a smooth connection on the vector bundle $F\to X$, the covariant derivative D_uA for A a smooth section of F and u a vector field on X is defined to be DA_*u [2]. If the connection is linear, the exterior derivative of a smooth F-valued r-form M on X (i.e. a smooth antisymmetric section of $L^r(TX; F)\to X$) is the smooth F-valued $(r+1)$-form dM on X defined by

$$dM(u_0, u_1, \cdots, u_r) = \sum_{i=0}^{r}(-1)^i D_{u_i}M(u_0, \cdots, \hat{u}_i, \cdots, u_r) + \sum_{i<j}(-1)^{i+j}M([u_i, u_j], \cdots, \hat{u}_i, \cdots, \hat{u}_j, \cdots, u_r),$$

where u_i are smooth vector fields on X. (Smoothness of the Christoffel component $\Gamma(x)(\_, \_) = \omega(x, \_)\_$ implies dM is a smooth section.)

For the case $r = 1$ the above equation reduces to

$$dM(u, v) = D_uMv - D_vMu - M[u, v]. \tag{3}$$

In terms of a local chart, M is represented by a smooth map

$$m\colon U \to L(\boldsymbol{B}, \boldsymbol{F}),$$

and dM is represented by the smooth map $U \to L^2(\boldsymbol{B}; \boldsymbol{F})$ defined by

$$dm(x)(\lambda, \mu) = m'(x)(\lambda)\mu - m'(x)(\mu)\lambda + \omega(x, m(x)\mu)\lambda - \omega(x, m(x)\lambda)\mu. \tag{4}$$

Now the connection form V of a smooth connection on $E_0 \subset E$ is locally the smooth map $U \times \boldsymbol{E}_U \to L(\boldsymbol{B} \times \boldsymbol{E}, \boldsymbol{E})$ defined by $V(x, a)(\lambda, b) = b + \omega(x, a)\lambda$. Using the Berwald connection on $VE | E_0 \to E_0$, (4) implies that the curvature form dV is locally

$$\begin{aligned} dV(x, a)((\lambda, b), (\mu, c)) = \omega'(x, a)(\lambda, -\omega(x, a)\lambda)\mu \\ - \omega'(x, a)(\mu, -\omega(x, a)\mu)\lambda, \end{aligned} \tag{5}$$

where $(x, a) \in U \times \boldsymbol{E}_U$. Clearly if either of (λ, b) or (μ, c) is vertical, i.e. if $\lambda = 0$ or $\mu = 0$, then the right side is 0. Hence

LEMMA. *The curvature form dV is horizontal, i.e. $dV(A,B) = dV(HA, HB)$ for all A, $B \in T_eE$, $e \in E_0$.*

THEOREM (STRUCTURE EQUATION). *If A and B denote vector fields on E_0, then $dV(A, B) = -V[HA, HB]$.*

PROOF. Immediate from (3) and the lemma.

An immediate consequence of the structure equation is the following generalization of a result of Sasaki for Riemannian connections [**4**, p. 343].

COROLLARY. *$HE \subset TE$ (over E_0) is integrable iff $dV = 0$.*

4. **The curvature tensor field.** The smooth isomorphisms $VE \approx p^{-1}E$ and $HE \approx p^{-1}TX$ are fibre-wise given by $r_e\colon (VE)_e \approx E_x$ and $p_*(e)\colon (HE)_e \approx T_xX$ (for $pe = x$). Their inverses are the vertical and horizontal lift maps [**2**], denoted $f_e^V = r_e^{-1}(f) \in (VE)_e$ and $u_e^H = p_*(e)^{-1}(u) = W_e(u) \in (HE)_e$, where $f \in E_x$ and $u \in T_xX$. These maps induce isomorphisms

$$L^2((HE)_e; (VE)_e) \approx L^2(T_xX; E_x) \qquad \text{for } e \in E_0 \qquad \text{with } pe = x.$$

Since each $dV(e)$ is horizontal, it corresponds to an antisymmetric map $R_x(\_, \_)e \in L^2(T_xX; E_x)$ by the above isomorphism. That is, it is defined as $R_x(u, v) = r(e)dV(u_e^H, v_e^H)$, or locally as

$$\begin{aligned} R(x)(\lambda, \mu)a &= \text{the right side of (5), which by [3, Proposition 11, p. 8] is} \\ &= \partial_1\omega(x, a)(\lambda)\mu - \partial_1\omega(x, a)(\mu)\lambda \\ &\quad + \partial_2\omega(x, a)(\omega(x, a)\mu)\lambda - \partial_2\omega(x, a)(\omega(x, a)\lambda)\mu. \end{aligned}$$

(This is equation (28) of [**1**, p. 138], with $\theta^h = dx^h$.)

The totality of maps $R_x(\_,\ \_)\_ : T_xX \times T_xX \times (E_x \cap E_0) \to E_x$ is the *curvature tensor field*. It does not in general define a tensor field on X in the usual sense of $x \mapsto R_x$ being a section of a vector bundle over X, because of lack of linearity in the e-slot. But if the connection on E_0 is homogeneous, then the maps $R_x(u, v)\_ : E_x - 0 \to E_x$ are smooth and homogeneous. If the connection is linear, then these maps are continuous linear, and $x \mapsto R_x$ defines a smooth section of $L^3(TX, TX, E; E) \to X$, which is the usual tensor field on X (because $\partial_2\omega(x, a)(c)\lambda = \omega(x, c)\lambda$ and $\partial_1\omega(x, a)(\lambda)\mu = \Gamma'(x)(\lambda)(a, \mu)$).

REMARK. The structural equation and the definition of D yield the following equation for the curvature tensor field of a general nonlinear connection:

$$R(u, v)e = -D[u^H, v^H]e \qquad \text{for all } e \in E_0,$$

where u, v are vector fields on X. (It is due to Dombrowski [2, p. 78 in the linear case.)

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