# ON THE BOUNDEDNESS OF AUTOMORPHIC FORMS

DAVID DRASIN AND C. J. EARLE[1]

1. **Introduction.** Let $\Gamma$ be a Fuchsian group acting on the unit disk $\Delta = \{z \in \boldsymbol{C}: |z| < 1\}$, and let $q \geqq 2$ be a fixed integer. The holomorphic function $\phi$ on $\Delta$ is called an *automorphic form* of weight $(-2q)$ if

$$\phi(Az)A'(z)^q = \phi(z), \qquad z \in \Delta, \quad A \in \Gamma. \tag{1}$$

Let $\omega$ be a Poincaré normal polygon for $\Gamma$ in $\Delta$. With Bers [2], we call the automorphic form $\phi$ *integrable* if

$$\|\phi\| = \iint_\omega |\phi(z)|(1 - |z|^2)^{q-2}dxdy < \infty \tag{2}$$

and *bounded* if

$$\|\phi\|_* = \sup\{|\phi(z)|(1 - |z|^2)^q : z \in \omega\} < \infty. \tag{3}$$

$A_q(\Gamma)$ is the Banach space of integrable forms normed by (2), and $B_q(\Gamma)$ is the Banach space of bounded forms, normed by (3). If $\Gamma$ contains only the identity map, we write $A_q$ and $B_q$ for $A_q(\Gamma)$ and $B_q(\Gamma)$.

We shall prove

THEOREM 1. *If $\Gamma$ is finitely generated, then $A_q(\Gamma) \subset B_q(\Gamma)$.*

THEOREM 2. *If $A_q(\Gamma) \subset B_q(\Gamma)$, then the inclusion map is continuous.*

Theorem 2 is a consequence of the closed graph theorem, and Theorem 1 is a consequence of this

LEMMA. *If $\Gamma$ is finitely generated and of the second kind, then there is an integrable form $\phi \in A_q(\Gamma)$ with the following properties:*
(a) $\sup\{|\phi(z)| : z \in \omega\} < \infty$.
(b) *If $\psi \in A_q(\Gamma)$, then $\psi(z) = f(z)\phi(z)$, where $f \in A_q$.*

Since $\omega$ has finite (noneuclidean) area if and only if $\Gamma$ is a finitely generated group of the first kind [2], [6], our theorems are generalizations of the well-known fact that $A_q(\Gamma) = B_q(\Gamma)$ when $\omega$ has finite area.

2. **Proof of the Lemma.** It is well known [2], [6] that to each finitely generated Fuchsian group $\Gamma$ of the second kind there corresponds a compact bordered Riemann surface $\overline{X}$ with interior $X$ such that

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$\Delta/\Gamma \subset X$, the set $X-\Delta/\Gamma$ is finite, and the natural map $\pi: \Delta \to \Delta/\Gamma$ is branched over only a finite number of points. Let $S=\{p_1, \cdots, p_n\}$ be the set of points $p \in X$ such that either $p \notin \Delta/\Gamma$ or the map $\pi$ is branched over $p$, and let $X'=X-S$. For $p \in S$, we set $l(p)=\infty$ if $p \notin X$ and $l(p)=n$ if the map $\pi$ has a branch point of order $n-1$ over $p$.

If $1<l(p)<\infty$, then $p=\pi(z_0)$, where $z_0 \in \Delta$ is the fixpoint of some $A \in \Gamma$ of order $n=l(p)$. A local parameter at $p$ is given by

$$\zeta(z) = (z-z_0)^n(1-\bar{z}_0 z)^{-n}.$$

If $l(p)=\infty$, then there is a parabolic transformation $A \in \Gamma$ of the form

$$(Az-a)^{-1} = (z-a)^{-1}+c, \qquad a \in \partial\Delta,$$

such that the function

$$\zeta(z) = \exp(\pi i(z+a)/ac(z-a)), \qquad z \in \Delta,$$

for $z$ close to $a$, defines a local parameter in a punctured neighborhood of $p$.

Using these parameters at the points of $S$, one finds that every meromorphic differential of dimension $q$ on $X$ which lifts to a function $\phi \in A_q(\Gamma)$ is a multiple of the divisor

$$D = -\sum_{p \in S} [q(1-1/l(p))]p.$$

Here $[x]$ denotes the greatest integer not exceeding $x$, with the understanding that $[q(1-1/\infty)]=q-1$.

By Abel's theorem there is a meromorphic first order differential $\alpha$ on the double of $\overline{X}$ which is analytic and nonzero in $\overline{X}$. Furthermore, there is a function $g$ analytic in $X$ and continuous in $\overline{X}$ which has at each $p \in S$ a zero of order $[q(1-1/l(p))]$ and no other zeros in $\overline{X}$. Then $\alpha^q/g$ is a meromorphic differential of dimension $q$ on $X$ with divisor $D$. This differential lifts to a holomorphic function $\phi$ on $\Delta$ which satisfies (1). It is easy to verify that $\phi$ is bounded in the fundamental polygon $\omega$. Hence $\|\phi\|<\infty$ and $\phi \in A_q(\Gamma)$.

On the other hand, each $\psi \in A_q(\Gamma)$ determines a meromorphic differential $\psi(z)dz^q$ on $X$ which is a multiple of the divisor $D$ of $\alpha^q/g$. This means that $\psi(z)=f(z)\phi(z)$, where $f(z)$ is the lift to $\Delta$ of an analytic function on $X$. (In particular, $f(Az)=f(z)$ for all $A$ in $\Gamma$.) We shall complete the proof of the Lemma by showing that $f \in A_q$.

To this end we choose a compact neighborhood $K \subset X$ of $S$, and we decompose $\Delta$ into the $\Gamma$-invariant sets $\Delta_1=\pi^{-1}(K)$ and $\Delta_2=\Delta-\Delta_1$. It is clear that

$$\iint_{\Delta_1} |f(z)|(1-|z|^2)^{q-2}dxdy < \infty \tag{4}$$

because the integrand is bounded in $\Delta_1$. On the other hand, if we put $\omega_2 = \omega \cap \Delta_2$, then

$$\sup\left\{\sum_{A\in\Gamma} |A'(z)|^q : z \in \omega_2\right\} = M < \infty,$$

and, since $\phi$ is continuous and nonzero in the closure of $\omega_2$,

$$\inf\{|\phi(z)| : z \in \omega_2\} = \delta > 0.$$

Thus,

$$\iint_{\Delta_2} |f(z)|(1-|z|^2)^{q-2}dxdy$$

$$= \sum_{A\in\Gamma}\iint_{A\omega_2} |f(z)|(1-|z|^2)^{q-2}dxdy$$

$$= \sum_{A\in\Gamma}\iint_{\omega_2} |f(A\zeta)|(1-|A\zeta|^2)^{q-2}|A'(\zeta)|^2 d\xi d\eta$$

$$= \sum_{A\in\Gamma}\iint_{\omega_2} |f(\zeta)|(1-|\zeta|^2)^{q-2}|A'(\zeta)|^q d\xi d\eta$$

$$\leqq M\delta^{-1}\iint_{\omega_2} |f(\zeta)\phi(\zeta)|(1-|\zeta|^2)^{q-2}d\xi d\eta \leqq M\delta^{-1}\|\psi\|.$$

Combining this inequality with (4), we find that $\|f\| < \infty$. The Lemma is proved.

3. **Proof of Theorem 1.** We may assume that $\Gamma$ is of the second kind, since otherwise the theorem is well known. By the Lemma, if $\psi \in A_q(\Gamma)$, then $\psi = f\phi$, with $f \in A_q$ and $\phi$ bounded in $\omega$. Choose $M$ so that $|\phi(z)| \leqq M$, $z \in \omega$. Then

$$\|\psi\|_* = \sup\{|\psi(z)|(1-|z|^2)^q : z \in \omega\} \leqq M\|f\|_*.$$

But $\|f\|_*$ is finite because $A_q$ is contained in $B_q$ by [2, p. 199]. The theorem is proved.

4. **Proof of Theorem 2.** By assumption, the injection $i$: $A_q(\Gamma) \to B_q(\Gamma)$ is defined on all of $A_q(\Gamma)$. Since for each $z \in \Delta$ the map $\phi \to \phi(z)$ is continuous on both $A_q(\Gamma)$ and $B_q(\Gamma)$, the graph of $i$ is closed. Hence $i$ is a continuous map.

5. **Remarks.** (i) Let $\Gamma$ be finitely generated and of the second kind. Let $E_q$ be the Banach subspace of $A_q$ consisting of those $f$ such that $f(Az)=f(z)$ for all $A\in\Gamma$. By the Lemma, the map $f\to f\phi$ is a continuous bijective map from $E_q$ to $A_q(\Gamma)$. Hence, by the open mapping theorem it is an isomorphism.

(ii) We are unable to prove that $A_q(\Gamma)\subset B_q(\Gamma)$ for arbitrary $\Gamma$. However, it is true that $A_q(\Gamma)\cap B_q(\Gamma)$ is always dense in $A_q(\Gamma)$. (This observation has been made independently by L. Bers.) To prove this, we note first that polynomials are dense in $A_q$, as an immediate consequence of [3, Lemma 1]. But a theorem of Bers [2], [4] asserts that the Poincaré series

$$\Theta\Phi(z) = \sum_{A\in\Gamma} \Phi(Az)A'(z)^q$$

defines a continuous map of $A_q$ onto $A_q(\Gamma)$. Hence the functions $\Theta P$, $P$ a polynomial, are dense in $A_q(\Gamma)$. Finally, each such function belongs to $B_q(\Gamma)$ by Godement's theorem on the boundedness of Poincaré series [5, Theorem 5 bis]; [1].

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Purdue University and
Cornell University