AN EXTENSION THEOREM FOR OBTAINING MEASURES 
ON UNCOUNTABLE PRODUCT SPACES

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Abstract. Several theorems are known for extending consistent families of measures to an inverse limit or product space [1]. In this paper the notion of a consistent family of measures is generalized so that, as with general product measures [2], the spaces are not required to be of unit measure or even \(\sigma\)-finite. The general extension problem may be separated into two parts, from finite to countable product spaces and from countable to uncountable product spaces. The first of these is discussed in [3]. The present paper concentrates on the second. The ultimate virtual identity of sets is defined and used as a key part of the generalization and nilsets similar to those of general product measures [2] are introduced to assure the measurability of the fundamental covering family. To exemplify the extension process, it is applied to product measures to obtain a general product measure. The paper is presented in terms of outer measures and Carathéodory measurability; however, some of the implications in terms of measure algebras should be obvious.

Introduction. An uncountable product space \(X = \prod_{i \in I} X_i\) is given with a family \(\mathcal{D}\) of countable subsets of the index set \(I\). To each such subset \(\tau \in \mathcal{D}\) there is an associated outer measure \(\mu_{\tau}\) on the countable product space \(X^\tau = \prod_{i \in \tau} X_i\). How the measures \(\mu_{\tau}\) are obtained is not of interest here, but to keep the complete extension problem in mind we might think that \(\mu_{\tau}\) is obtained as in [3] by extending a regular conditional measure system onto \(X^\tau\). The problem we are concerned with here is that of stipulating conditions on the system of measures \(\mu_{\tau}\) that allow their extension to an outer measure \(\mu\) on \(X\) having properties reflecting their own.

To proceed with the problem we need to agree on some notation. If \(A \subseteq X\) and \(\sigma\) is any subset of \(I\), let \(A_{\sigma}\) be the projection of \(A\) onto the space \(X_{\sigma}\), and if \(a \subseteq X_{\sigma}\) let \(a^*\) be the cylinder in \(X\) over \(a\). The symmetric difference between two sets \(A\) and \(B\) will be denoted by \(A \Delta B\), and for \(\tau \in \mathcal{D}\), its complement \(I - \tau\) relative to \(I\) will be denoted by \(\tau'\).

The family \(\mathcal{D}\) is said to be comprehensive when each countable subset \(\sigma\) of \(I\) is contained in some element \(\tau\) of \(\mathcal{D}\) (i.e. \(\sigma \subseteq \tau \in \mathcal{D}\)). In the remainder of the paper we assume that \(\mathcal{D}\) is comprehensive.

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1. The covering family. The existence of the covering family \( \mathcal{F} \) described below requires that the measures \( \mu_r \) be rather congenially related. It is in this definition that the notion of a consistent family of measures is being generalized. To give the definition, we must first introduce two families of sets, the first of which is a traditional part of the extension process and the second of which is reminiscent of product measures. For \( \tau \in \mathcal{D} \) let \( \mathcal{M}_r = \{ \beta \subset X^\tau : \beta \) is Carathéodory measurable with respect to \( \mu_r \} \) and

\[
\mathcal{B}_r = \left\{ \beta \subset X^{\tau'} : \beta = \prod_{i \in \tau'} \beta_i \text{ where } \beta_i \subset X_i \text{ for each } i \in \tau', \text{ and} \right\}
\]

\[
(\alpha \times \beta)_\sigma \in \mathcal{M}_\sigma \text{ for each } \alpha \in \mathcal{M}_r \text{ and each } \sigma \in \mathcal{D} \text{ such that } \tau \subset \sigma. \]

For a covering family \( \mathcal{F} \) we require a family of subsets of \( X \) such that for each \( A \in \mathcal{F} \) there exists \( \tau \in \mathcal{D} \) with \( A = \alpha \times \beta \) for some \( \alpha \in \mathcal{M}_r \) and \( \beta \in \mathcal{B}_r \), and \( \mu_\sigma(A_\sigma) = \mu_r(\alpha) \) whenever \( \sigma \in \mathcal{D} \) and \( \tau \subset \sigma \). (Note here, as a consequence of the definition of \( \mathcal{B}_r \), that also \( \mu_\sigma \in \mathcal{M}_\sigma \).) Further restrictions will subsequently be placed on \( \mathcal{F} \).

In the above, \( \beta \) is in some sense a set of unit measure relative to \( \alpha \) and \( A \) is the analogue of a classical cylinder set. On \( \mathcal{F} \) we can now define a function \( \Psi \) by means of \( \Psi(A) = \mu_r(A) \) where \( A = \alpha \times \beta \), \( \alpha \in \mathcal{M}_r \), \( \beta \in \mathcal{B}_r \), \( \tau \in \mathcal{D} \), and \( \mu_\sigma(A_\sigma) = \mu_r(\alpha) \) whenever \( \sigma \in \mathcal{D} \) and \( \tau \subset \sigma \). Using \( \mu \) as a gauge and \( \mathcal{F} \) as a covering family, we generate an outer measure \( \Psi \) on \( X \) by taking \( \Psi(A) \), \( A \subset X \), to be the infimum of numbers of the form \( \sum_{n \in \mathbb{N}} \mu(B) \) where \( \mathcal{G} \) is a countable subfamily of \( \mathcal{F} \) which covers \( A \). We come now to our first

**Theorem 1.1.** If \( A \in \mathcal{F} \) then \( \Psi(A) = \mu(A) \).

**Proof.** If \( \mathcal{G} \) is a countable subfamily of \( \mathcal{F} \) such that \( A \subset \bigcup \mathcal{G} \), then for each \( B \in \mathcal{G} \cup \{ A \} \) let \( \tau_B \) be such a member of \( \mathcal{D} \) that \( \mu(B) = \mu_{\tau_B}(B_{\tau_B}) \) and let \( \sigma \) be such a member of \( \mathcal{D} \) that

\[
\bigcup_{B \in \mathcal{G} \cup \{ A \}} \tau_B \subset \sigma.
\]

Thus \( \mu(B) = \mu_\sigma(B_\sigma) \) for each \( B \in \mathcal{G} \cup \{ A \} \) and

\[
\mu(A) = \mu_\sigma(A_\sigma) \leq \sum_{B \in \mathcal{G}} \mu_\sigma(B_\sigma) = \sum_{B \in \mathcal{G}} \mu(B),
\]

from which we may conclude that \( \Psi(A) = \mu(A) \).

The first basic assumption to be made about \( \mathcal{F} \) is that if \( \tau \in \mathcal{D} \), \( \alpha \in \mathcal{M}_r \), and \( A \in \mathcal{F} \), then \( A \alpha^* \in \mathcal{F} \) and \( A - \alpha^* \in \mathcal{F} \). This leads to our second theorem.
Theorem 1.2. If $\tau \in \mathcal{D}$ and $\alpha \in \mathcal{M}_\tau$, then $\alpha^*$ is $\Psi$ measurable.

Proof. Since $\mathcal{F}$ is the covering family for $\Psi$, it is sufficient to show that $\mu(T) = \mu(T\alpha^*) + \mu(T - \alpha^*)$ for each $T \in \mathcal{F}$. Suppose then that $T \in \mathcal{F}$ and let $\sigma$ be such a member of $\mathcal{D}$ that $\tau \subset \sigma$, each of the sets $(T\alpha^*)_\sigma$, $(T - \alpha^*)_\sigma$ and $T_\sigma$ belong to $\mathcal{M}_\sigma$ and $\mu(T\alpha^*) = \mu_\sigma((T\alpha^*)_\sigma)$, $\mu(T - \alpha^*) = \mu_\sigma((T - \alpha^*)_\sigma)$ and $\mu(T) = \mu_\sigma(T_\sigma)$. Then we have

$$\mu(T) = \mu_\sigma(T_\sigma) = \mu_\sigma((T\alpha^*)_\sigma) + \mu_\sigma((T - \alpha^*)_\sigma) = \mu(T\alpha^*) + \mu(T - \alpha^*),$$

which completes the proof.

The next basic assumption to be made about $\mathcal{F}$ is that it be intersective, i.e. if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. With this we come to

2. Nilsets and ultimate virtual identity. Somewhat parallel to the definition of nilsets given in [2] we define a family of nilsets $\mathcal{N}$ by

$$\mathcal{N} = \left\{ N : N = \bigcup_{i \in I} n_i^* \text{ where } n_i \subset X_i, A - N \in \mathcal{F} \right\}.$$ 

and $\mu(A - N) = \mu(A)$ whenever $A \in \mathcal{F}$.

Two members $A$ and $B$ of $\mathcal{F}$ are called ultimately virtually identical (u.v.i.) provided for some $\tau \in \mathcal{D}$, $\bigcup_{i \in I^\tau} (A_{(i)} \Delta B_{(i)})^* \in \mathcal{N}$. Since $\mathcal{D}$ is comprehensive, $\mathcal{D}$ is a directed set. The term ultimate refers to ultimate in the sense of the direction on $\mathcal{D}$ and virtual identity refers to differences that amount to a nilset.

We now introduce a rather strong but natural assumption concerned with the idea that if two members of $\mathcal{F}$ have much in common, then they are u.v.i. Specifically, our third assumption is that $\mathcal{F}$ satisfies the condition that if $A \in \mathcal{F}$, $B \in \mathcal{F}$, and $\mu(AB) > 0$ then $A$ and $B$ are u.v.i.

At this point we modify our measure $\Psi$ by requiring that members of $\mathcal{N}$ have zero measure. We define $\phi$ to be the function on the subsets of $X$ such that

$$\phi(A) = \inf_{N \in \mathcal{N}} \Psi(A - N)$$

whenever $A \subset X$.

As our fourth and final assumption about our system of measures we ask that $\mathcal{N}$ be closed to countable unions. Then $\phi$ turns out to be a measure which agrees with $\Psi$ on $\mathcal{F}$ and may be generated by the covering family $\mathcal{F} \cup \mathcal{N}$ and a gauge $\mu'$ which equals $\mu$ on $\mathcal{F}$ but is zero on $\mathcal{N}$. The point to the above modification of $\Psi$ is to achieve the measurability of the members of $\mathcal{F}$. This brings us to
Theorem 2.1. If $A \in \mathcal{F}$, then $A$ is $\phi$ measurable and $\phi(A) = \Psi(A) = \mu(A)$ and if $N \in \mathcal{N}$, then $\phi(N) = 0$.

Proof. In view of the definitions of $\mathcal{N}$ and $\phi$ and Theorem 1.1 it is evident that $\phi(A) = \Psi(A) = \mu(A)$ and that $\phi(N) = 0$. To see that $A$ is $\phi$ measurable it is only necessary to check that $\phi(T) = \phi(TA) + \phi(T - A)$ for each $T \in \mathcal{F} \cup \mathcal{N}$, since $\mathcal{F} \cup \mathcal{N}$ is a covering family for $\phi$. The above equation is trivially satisfied when $T \in \mathcal{F}$ so let us suppose that $T \in \mathcal{N}$. Now if $\mu(AT) > 0$, let $\sigma_1$ be a member of $\mathcal{D}$ for which $N_1 = \bigcup_{i \in \sigma_1} (T_{[i]} \setminus A_{[i]})^* \in \mathcal{N}$. In view of the properties of $\mathcal{F}$ discussed at the beginning of §1 we can take $\sigma_2 \in \mathcal{D}$ large enough that $A = A_{\sigma_2} \times \beta$ where $A_{\sigma_2} \in \mathcal{M}_{\sigma_2}$ and $\beta \in \mathcal{G}_{\sigma_2}$. Now let $T$ be such a member of $\mathcal{D}$ that $\sigma_1 \subseteq T$ and $\sigma_2 \subseteq T$, and let $B = (A_i)^*$. Then check that $A = B \cap \bigcap_{i \in T} A_{[i]}^*$ and with the aid of Theorem 1.2 infer that $B$ is $\Psi$ measurable and hence also $\phi$ measurable. Let $N = \bigcup_{i \in T} (T_{[i]} \setminus A_{[i]})^*$ and note that $\tau' \subseteq \sigma_1'$ and hence $N \subseteq N_1$. In view of this and the fact that $\phi(N_1) = 0$, we conclude that $\phi(N) = 0$ also.

Now, since $A = B \cap \bigcap_{i \in \tau} A_{[i]}^*$, we have

$$T - A = T - B \cup \bigcup_{i \in \tau'} (T - A_{[i]}^*) \subseteq T - B \cup \bigcup_{i \in \tau'} (T_{[i]}^* - A_{[i]}^*)$$

$$= T - B \cup \bigcup_{i \in \tau'} (T_{[i]} - A_{[i]})^* \subseteq T - B \cup \bigcup_{i \in \tau'} (T_{[i]} \setminus A_{[i]})^*$$

from which we infer that $T - A \subseteq T - B \cup N$. Noting further that $A \subseteq B$, we conclude

$$\phi(T) \leq \phi(TA) + \phi(T - A) \leq \phi(TB) + \phi(T - B \cup N)$$

$$\leq \phi(TB) + \phi(T - B) + \phi(N) = \phi(T) + 0$$

and

$$\phi(T) = \phi(TA) + \phi(T - A).$$

Now, if $\mu(\mathcal{A}T) = 0$, then

$$\phi(T) \leq \phi(TA) + \phi(T - A) \leq 0 + \phi(T - A) \leq \phi(T).$$

Hence $\phi(T) = \phi(TA) + \phi(T - A)$ whenever $T \in \mathcal{F}$ and the proof is complete.

3. An application to product measures. Suppose that for each $i \in I$, $\lambda_i$ is an arbitrary (outer) measure on $X_i$ and that $\mathcal{D}$ is the family of countable subsets of $I$. Then $\mathcal{D}$ is clearly comprehensive. Now, for $\tau \in \mathcal{D}$, $\tau = \{i_1, i_2, \ldots, i_r, \ldots\}$, we can define a regular conditional measure system $\nu_\tau$ on $X_\tau$ by taking $\nu_0 = \lambda_{i_1}$ and $\nu_\tau(x, \cdot) = \lambda_{i_{r+1}}(\cdot)$. For
each \( x \in \prod_{i=1}^{r} X_i \). Then, by the construction in [3] we obtain from this regular conditional measure system a measure \( \mu_r \) on \( X^r \) for which

\[
\mu_r(\beta) = \prod_{r=1}^{\infty} \lambda_{i_r}(\beta_r),
\]

where \( \beta = \prod_{r=1}^{\infty} \beta_r \) and for each \( r, \beta_r \) is a \( \lambda_r \) measurable subset of \( X_r \) and \( \prod_{r=1}^{\infty} \lambda_{i_r}(\beta_r) < \infty \).

For the system \( \mu_r, r \in \mathbb{D} \), it can be shown that we can take

\[
G = \{ A: \text{for some } r \in \mathbb{D}, A = \alpha \times \beta \text{ where } \alpha \text{ is a } \mu_r \text{ measurable set, } \mu_r(\alpha) < \infty, \text{ and } \beta = \prod_{i \in \tau'} \beta_i \text{ where for each } i \in \tau', \beta_i \text{ is a } \lambda_i \text{ measurable subset of } X_i \text{ and (1) } \mu_r(\alpha) = 0 \text{ or } \mu_r(\alpha) > 0 \text{ and } \lambda_i(\beta_i) = 1 \text{ for each } i \in \tau' \}
\]

and

\[
\mathcal{G} = \left\{ N: N = \bigcup_{i \in I} n_i \text{ where } \lambda_i(n_i) = 0 \text{ for each } i \in I \right\}.
\]

It is clear that if \( A \in G, B \in G \) and \( \mu(AB) > 0 \) then for some \( r \in \mathbb{D} \),

\[
\lambda_i((AB)_{i_i}) = 1 \text{ for each } i \in \tau'
\]

and consequently \( A \) and \( B \) are u.v.i. Furthermore, \( AB \in G \). If \( \mu(AB) = 0 \), then for some \( \sigma \in \mathbb{D}, \mu_{i}(((AB)_{i}) = 0 \) and again we see that \( AB \in G \). Hence \( G \) is intersective. Noting finally that \( \mathcal{G} \) is closed to countable unions we see that all of our assumptions are met and we obtain the measure \( \phi \) on \( X \) with the properties stated in Theorem 2.1. This measure is essentially the general product measure of [2]. By breaking the extension into two parts, finite to countable, and countable to uncountable, the end result is reached more simply here than it is in [2].

References


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