AN EXTENSION THEOREM FOR OBTAINING MEASURES ON UNCOUNTABLE PRODUCT SPACES

E. O. ELLIOTT

Abstract. Several theorems are known for extending consistent families of measures to an inverse limit or product space [1]. In this paper the notion of a consistent family of measures is generalized so that, as with general product measures [2], the spaces are not required to be of unit measure or even σ-finite. The general extension problem may be separated into two parts, from finite to countable product spaces and from countable to uncountable product spaces. The first of these is discussed in [3]. The present paper concentrates on the second. The ultimate virtual identity of sets is defined and used as a key part of the generalization and nilsets similar to those of general product measures [2] are introduced to assure the measurability of the fundamental covering family. To exemplify the extension process, it is applied to product measures to obtain a general product measure. The paper is presented in terms of outer measures and Carathéodory measurability; however, some of the implications in terms of measure algebras should be obvious.

Introduction. An uncountable product space $X=\prod_{i\in I} X_i$ is given with a family $\mathfrak{D}$ of countable subsets of the index set I. To each such subset $\tau\in\mathfrak{D}$ there is an associated outer measure μ_τ on the countable product space $X^\tau=\prod_{i\in\tau} X_i$. How the measures μ_τ are obtained is not of interest here, but to keep the complete extension problem in mind we might think that μ_τ is obtained as in [3] by extending a regular conditional measure system onto X^τ. The problem we are concerned with here is that of stipulating conditions on the system of measures μ_τ that allow their extension to an outer measure μ on X having properties reflecting their own.

To proceed with the problem we need to agree on some notation. If $A\subset X$ and σ is any subset of I, let A_σ be the projection of A onto the space X^σ, and if $a\subset X^\sigma$ let a^* be the cylinder in X over a. The symmetric difference between two sets A and B will be denoted by $A\triangle B$, and for $\tau\in\mathfrak{D}$, its complement $I-\tau$ relative to I will be denoted by τ'.

The family $\mathfrak{D}$ is said to be comprehensive when each countable subset σ of I is contained in some element τ of $\mathfrak{D}$ (i.e. $\sigma\subset\tau\in\mathfrak{D}$). In the remainder of the paper we assume that $\mathfrak{D}$ is comprehensive.

Received by the editors October 14, 1966 and, in revised form, May 5, 1967.

1. **The covering family.** The existence of the covering family $\mathfrak{F}$ described below requires that the measures μ_τ be rather congenially related. It is in this definition that the notion of a consistent family of measures is being generalized. To give the definition, we must first introduce two families of sets, the first of which is a traditional part of the extension process and the second of which is reminiscent of product measures. For $\tau \in \mathfrak{D}$ let $\mathfrak{M}_\tau = \{\beta \subset X^\tau : \beta$ is Carathéodory measurable with respect to $\mu_\tau\}$ and

$$\mathcal{B}_\tau = \left\{\beta \subset X^{\tau'} : \beta = \prod_{i \in \tau'} \beta_i \text{ where } \beta_i \subset X_i \text{ for each } i \in \tau', \text{ and} \right.$$

$$\left. (\alpha \times \beta)_\sigma \in \mathfrak{M}_\sigma \text{ for each } \alpha \in \mathfrak{M}_\tau \text{ and each } \sigma \in \mathfrak{D} \text{ such that } \tau \subset \sigma \right\}.$$

For a covering family $\mathfrak{F}$ we require a family of subsets of X such that for each $A \in \mathfrak{F}$ there exists $\tau \in \mathfrak{D}$ with $A = \alpha \times \beta$ for some $\alpha \in \mathfrak{M}_\tau$ and $\beta \in \mathcal{B}_\tau$, and $\mu_\sigma(A_\sigma) = \mu_\tau(\alpha)$ whenever $\sigma \in \mathfrak{D}$ and $\tau \subset \sigma$. (Note here, as a consequence of the definition of $\mathcal{B}_\tau$, that also $A_\sigma \in \mathfrak{M}_\sigma$.) Further restrictions will subsequently be placed on $\mathfrak{F}$.

In the above, β is in some sense a set of unit measure relative to α and A is the analogue of a classical cylinder set. On $\mathfrak{F}$ we can now define a function μ by means of $\mu(A) = \mu_\tau(\alpha)$ where $A = \alpha \times \beta$, $\alpha \in \mathfrak{M}_\tau$, $\beta \in \mathcal{B}_\tau$, $\tau \in \mathfrak{D}$, and $\mu_\sigma(A_\sigma) = \mu_\tau(\alpha)$ whenever $\sigma \in \mathfrak{D}$ and $\tau \subset \sigma$. Using μ as a gauge and $\mathfrak{F}$ as a covering family, we generate an outer measure Ψ on X by taking $\Psi(A)$, $A \subset X$, to be the infimum of numbers of the form $\sum_{B \in \mathcal{G}} \mu(B)$ where $\mathcal{G}$ is a countable subfamily of $\mathfrak{F}$ which covers A. We come now to our first

THEOREM 1.1. *If* $A \in \mathfrak{F}$ *then* $\Psi(A) = \mu(A)$.

PROOF. If $\mathcal{G}$ is a countable subfamily of $\mathfrak{F}$ such that $A \subset \bigcup \mathcal{G}$, then for each $B \in \mathcal{G} \cup \{A\}$ let τ_B be such a member of $\mathfrak{D}$ that $\mu(B) = \mu_{\tau_B}(B_{\tau_B})$ and let σ be such a member of $\mathfrak{D}$ that

$$\bigcup_{B \in \mathcal{G} \cup \{A\}} \tau_B \subset \sigma.$$

Thus $\mu(B) = \mu_\sigma(B_\sigma)$ for each $B \in \mathcal{G} \cup \{A\}$ and

$$\mu(A) = \mu_\sigma(A_\sigma) \leqq \sum_{B \in \mathcal{G}} \mu_\sigma(B_\sigma) = \sum_{B \in \mathcal{G}} \mu(B),$$

from which we may conclude that $\Psi(A) = \mu(A)$.

The first basic assumption to be made about $\mathfrak{F}$ is that if $\tau \in \mathfrak{D}$, $\alpha \in \mathfrak{M}_\tau$, and $A \in \mathfrak{F}$, then $A\alpha^* \in \mathfrak{F}$ and $A - \alpha^* \in \mathfrak{F}$. This leads to our second theorem.

THEOREM 1.2. *If* $\tau \in \mathfrak{D}$ *and* $\alpha \in \mathfrak{M}_\tau$, *then* α^* *is* Ψ *measurable.*

PROOF. Since $\mathfrak{F}$ is the covering family for Ψ, it is sufficient to show that $\mu(T) = \mu(T\alpha^*) + \mu(T - \alpha^*)$ for each $T \in \mathfrak{F}$. Suppose then that $T \in \mathfrak{F}$ and let σ be such a member of $\mathfrak{D}$ that $\tau \subset \sigma$, each of the sets $(T\alpha^*)_\sigma$, $(T - \alpha^*)_\sigma$ and T_σ belong to $\mathfrak{M}_\sigma$ and $\mu(T\alpha^*) = \mu_\sigma((T\alpha^*)_\sigma)$, $\mu(T - \alpha^*) = \mu_\sigma((T - \alpha^*)_\sigma)$ and $\mu(T) = \mu_\sigma(T_\sigma)$. Then we have

$$\mu(T) = \mu_\sigma(T_\sigma) = \mu_\sigma((T\alpha^*)_\sigma) + \mu_\sigma((T - \alpha^*)_\sigma) = \mu(T\alpha^*) + \mu(T - \alpha^*),$$

which completes the proof.

The next basic assumption to be made about $\mathfrak{F}$ is that it be intersective, i.e. if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $AB \in \mathfrak{F}$. With this we come to

2. **Nilsets and ultimate virtual identity.** Somewhat parallel to the definition of nilsets given in [2] we define a family of nilsets $\mathfrak{N}$ by

$$\mathfrak{N} = \left\{ N : N = \bigcup_{i \in I} n_i^* \text{ where } n_i \subset X_i,\ A - N \in \mathfrak{F} \right.$$
$$\left. \text{and } \mu(A - N) = \mu(A) \text{ whenever } A \in \mathfrak{F} \right\}.$$

Two members A and B of $\mathfrak{F}$ are called ultimately virtually identical (u.v.i.) provided for some $\tau \in \mathfrak{D}$, $\bigcup_{i \in \tau'} (A_{\{i\}} \triangle B_{\{i\}})^* \in \mathfrak{N}$. Since $\mathfrak{D}$ is comprehensive, $\mathfrak{D}$ is a directed set. The term ultimate refers to ultimate in the sense of the direction on $\mathfrak{D}$ and virtual identity refers to differences that amount to a nilset.

We now introduce a rather strong but natural assumption concerned with the idea that if two members of $\mathfrak{F}$ have much in common, then they are u.v.i. Specifically, our third assumption is that $\mathfrak{F}$ satisfies the condition that if $A \in \mathfrak{F}$, $B \in \mathfrak{F}$, and $\mu(AB) > 0$ then A and B are u.v.i.

At this point we modify our measure Ψ by requiring that members of $\mathfrak{N}$ have zero measure. We define ϕ to be the function on the subsets of X such that

$$\phi(A) = \inf_{N \in \mathfrak{N}} \Psi(A - N)$$

whenever $A \subset X$.

As our fourth and final assumption about our system of measures we ask that $\mathfrak{N}$ be closed to countable unions. Then ϕ turns out to be a measure which agrees with Ψ on $\mathfrak{F}$ and may be generated by the covering family $\mathfrak{F} \cup \mathfrak{N}$ and a gauge μ' which equals μ on $\mathfrak{F}$ but is zero on $\mathfrak{N}$. The point to the above modification of Ψ is to achieve the measurability of the members of $\mathfrak{F}$. This brings us to

THEOREM 2.1. *If* $A \in \mathfrak{F}$, *then* A *is* ϕ *measurable and* $\phi(A) = \Psi(A) = \mu(A)$ *and if* $N \in \mathfrak{N}$, *then* $\phi(N) = 0$.

PROOF. In view of the definitions of $\mathfrak{N}$ and ϕ and Theorem 1.1 it is evident that $\phi(A) = \Psi(A) = \mu(A)$ and that $\phi(N) = 0$. To see that A is ϕ measurable it is only necessary to check that $\phi(T) = \phi(TA) + \phi(T - A)$ for each $T \in \mathfrak{F} \cup \mathfrak{N}$, since $\mathfrak{F} \cup \mathfrak{N}$ is a covering family for ϕ. The above equation is trivially satisfied when $T \in \mathfrak{N}$ so let us suppose that $T \in \mathfrak{F}$. Now if $\mu(AT) > 0$, let σ_1 be a member of $\mathfrak{D}$ for which $N_1 = \bigcup_{i \in \sigma_1'} (T_{\{i\}} \triangle A_{\{i\}})^* \in \mathfrak{N}$. In view of the properties of $\mathfrak{F}$ discussed at the beginning of §1 we can take $\sigma_2 \in \mathfrak{D}$ large enough that $A = A_{\sigma_2} \times \beta$ where $A_{\sigma_2} \in \mathfrak{M}_{\sigma_2}$ and $\beta \in \mathfrak{B}_{\sigma_2}$. Now let τ be such a member of D that $\sigma_1 \subset \tau$ and $\sigma_2 \subset \tau$, and let $B = (A_\tau)^*$. Then check that $A = B \cap \bigcap_{i \in \tau'} A^*_{\{i\}}$ and with the aid of Theorem 1.2 infer that B is Ψ measurable and hence also ϕ measurable. Let $N = \bigcup_{i \in \tau'} (T_{\{i\}} \triangle A_{\{i\}})^*$ and note that $\tau' \subset \sigma_1'$ and hence $N \subset N_1$. In view of this and the fact that $\phi(N_1) = 0$, we conclude that $\phi(N) = 0$ also.

Now, since $A = B \cap \bigcap_{i \in \tau'} A^*_{\{i\}}$, we have

$$T - A = T - B \cup \bigcup_{i \in \tau'} (T - A^*_{\{i\}}) \subset T - B \cup \bigcup_{i \in \tau'} (T^*_{\{i\}} - A^*_{\{i\}})$$
$$= T - B \cup \bigcup_{i \in \tau'} (T_{\{i\}} - A_{\{i\}})^* \subset T - B \cup \bigcup_{i \in \tau'} (T_{\{i\}} \triangle A_{\{i\}})^*$$

from which we infer that $T - A \subset T - B \cup N$. Noting further that $A \subset B$, we conclude

$$\phi(T) \leqq \phi(TA) + \phi(T - A) \leqq \phi(TB) + \phi(T - B \cup N)$$
$$\leqq \phi(TB) + \phi(T - B) + \phi(N) = \phi(T) + 0$$

and

$$\phi(T) = \phi(TA) + \phi(T - A).$$

Now, if $\mu(AT) = 0$, then

$$\phi(T) \leqq \phi(TA) + \phi(T - A) \leqq 0 + \phi(T - A) \leqq \phi(T).$$

Hence $\phi(T) = \phi(TA) + \phi(T - A)$ whenever $T \in \mathfrak{F}$ and the proof is complete.

3. **An application to product measures.** Suppose that for each $i \in I$, λ_i is an arbitrary (outer) measure on X_i and that $\mathfrak{D}$ is the family of countable subsets of I. Then $\mathfrak{D}$ is clearly comprehensive. Now, for $\tau \in \mathfrak{D}$, $\tau = \{i_1, i_2, \cdots, i_r, \cdots\}$, we can define a regular conditional measure system ν_τ on X^τ by taking $\nu_0 = \lambda_{i_1}$ and $\nu_r(x, \cdot) = \lambda_{i_{r+1}}(\cdot)$ for

each $x \in \prod_{t=1}^{r} X_{i_t}$. Then, by the construction in [3] we obtain from this regular conditional measure system a measure μ_τ on X^τ for which

$$\mu_\tau(\beta) = \prod_{r=1}^{\infty} \lambda_{i_r}(\beta_r),$$

where $\beta = \prod_{r=1}^{\infty} \beta_r$ and for each r, β_r is a λ_{i_r} measurable subset of X_{i_r} and $\prod_{r=1}^{\infty} \lambda_{i_r}(\beta_r) < \infty$.

For the system μ_τ, $\tau \in \mathfrak{D}$, it can be shown that we can take

$$\mathfrak{F} = \{A: \text{for some } \tau \in \mathfrak{D},\ A = \alpha \times \beta \text{ where } \alpha \text{ is a } \mu_\tau \text{ measurable set, } \mu_\tau(\alpha) < \infty, \text{ and } \beta = \textstyle\prod_{i \in \tau'} \beta_i \text{ where for each } i \in \tau', \beta_i \text{ is a } \lambda_i \text{ measurable subset of } X_i \text{ and (1) } \mu_\tau(\alpha) = 0 \text{ or (2) } \mu_\tau(\alpha) > 0 \text{ and } \lambda_i(\beta_i) = 1 \text{ for each } i \in \tau'\}$$

and

$$\mathfrak{N} = \left\{N: N = \bigcup_{i \in I} n_i^* \text{ where } \lambda_i(n_i) = 0 \text{ for each } i \in I\right\}.$$

It is clear that if $A \in \mathfrak{F}$, $B \in \mathfrak{F}$ and $\mu(AB) > 0$ then for some $\tau \in \mathfrak{D}$,

$$\lambda_i((AB)_{\{i\}}) = 1 \qquad \text{for each } i \in \tau'$$

and consequently A and B are u.v.i. Furthermore, $AB \in \mathfrak{F}$. If $\mu(AB) = 0$, then for some $\sigma \in \mathfrak{D}$, $\mu_\sigma((AB)_\sigma) = 0$ and again we see that $AB \in \mathfrak{F}$. Hence $\mathfrak{F}$ is intersective. Noting finally that $\mathfrak{N}$ is closed to countable unions we see that all of our assumptions are met and we obtain the measure ϕ on X with the properties stated in Theorem 2.1. This measure is essentially the general product measure of [2]. By breaking the extension into two parts, finite to countable, and countable to uncountable, the end result is reached more simply here than it is in [2].

References

1. J. R. Choksi, *Inverse limits of measure spaces*, Proc. London Math. Soc. (3) **8** (1958), 321–342.

2. E. O. Elliott and A. P. Morse, *General product measures*, Trans. Amer. Math. Soc. (2) **110** (1964), 245–283.

3. E. O. Elliott, *Measures on countable product spaces*, Pacific J. Math. (to appear).

Bell Telephone Laboratories, Inc., Holmdel, New Jersey