A NOTE ON CONNECTEDNESS IM KLEINEN

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1. Introduction. In this paper S denotes a compact Hausdorff continuum. If $A \subset S$, $T(A)$ denotes the complement of the set of those points p of S for which there exists a subcontinuum W of S such that $y \in \operatorname{Int}(W)$ and $W \cap A = \varnothing$. Equivalently $p \in T(A)$ iff every subcontinuum that has p as an interior point intersects A nonvoidly. Basic properties of the set function T are to be found in [1].

The concepts of "locally connected" and "connected im kleinen" are taken as known. It is noted that S is connected im kleinen at a point p of S iff for every open set U that contains p, $p \notin T(\operatorname{Fr}(U))$ or, equivalently, for every closed $A \subset S$, $p \in A$ iff $p \in T(A)$. S is said to be semilocally connected [4] at a point p of S provided that for every open set U that contains p, there exists an open set V that contains p such that $V \subset U$ and $S - V$ has a finite number of components. S is semilocally connected at a point p of S iff $T(p) = p$ [3].

2. Decomposability and connectedness im kleinen. Let A and B be subsets of S, S is decomposable about A and B iff there exists subcontinua M and N of S such that $A \subset M - N$, $B \subset N - M$ and $S = M \cup N$.

THEOREM 1. *Let A and B be subsets of S. If S is decomposable about A and B then $A \cap T(B) = \varnothing = B \cap T(A)$.*

PROOF. This follows easily from the definitions.

THEOREM 2. *Let A and B be subcontinua of S. If $A \cap T(B) = \varnothing = B \cap T(A)$ then S is decomposable about A and B.*

PROOF. Since $A \cap T(B) = \varnothing = B \cap T(A)$ and A and B are compact continua, there exist continua W_A and W_B such that $A \subset \operatorname{Int}(W_A)$, $B \subset \operatorname{Int}(W_B)$, $W_A \cap B = \varnothing$ and $W_B \cap A = \varnothing$. Let $V_A = \operatorname{Int}(W_A) - W_B$ and $V_B = \operatorname{Int}(W_B) - W_A$. Let K_A be the component of $S - V_B$ that contains W_A and let K_B be the component of $S - V_A$ that contains W_B. Suppose that $S \neq K_A \cup K_B$. Then let $L = S - (K_A \cup K_B)$ and note $L \neq \varnothing$. Since $K_B \cup \bar{L} \subset S - V_A$ and K_B is a proper subset of $K_B \cup \bar{L}$, $K_B \cup \bar{L}$ is the union of two disjoint closed nonvoid sets P_A and P_B

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and K_B may be assumed to lie in P_B. Then $P_A \cup K_A$ is the union of two disjoint closed nonvoid sets Q_A and Q. K_A may be assumed to lie in Q_A. Thus $S = Q \cup (P_B \cup Q_A)$ which is a separation of S since $Q \subset P_A$.

Theorems 1 and 2 are generalizations of Theorems 5 and 6 of [2]. The proof of Theorem 2 is similar to the proof of Jones's Theorem 6.

THEOREM 3. *Let $A \subset S$ and a and b be points of S. If $T(A) = A$ and a and b lie in the same component K of $S - A$ then there exists a continuum W such that $\{a\} \cup \{b\} \subset W \subset S - A$.*

PROOF. For each point $x \in K$ there exists a continuum W_x such that $x \in \text{Int}(W_x)$ and $W_x \subset K$. Since $\{\text{Int}(W_x): x \in K\}$ covers K there exists a finite chain, $\text{Int}(W_1)$, $\text{Int}(W_1)$, $\cdots$, $\text{Int}(W_m)$ such that $a \in \text{Int}(W_1)$, $b \in \text{Int}(W_m)$ and $\text{Int}(W_j) \cap \text{Int}(W_{j+1}) \neq \varnothing$ for $j = 1, \cdots, m-1$. Let $W = W_1 \cup W_2 \cup \cdots \cup W_m$. Thus W is the desired continuum.

Theorem 3 is a generalization of Theorem 6.2 of [4].

THEOREM 4. *Let p be a point of S. If $T(p) = p$ and if, for every subcontinuum W of S that does not contain p, $p \notin T(W)$ then S is connected im kleinen at p.*

PROOF. Let U be an open set containing p. Since $T(p) = p$, S is semilocally connected at p and therefore there exists an open set $V \subset U$ containing p such that $S - V = W_1 \cup \cdots \cup W_m$ where each W_i is a subcontinuum of S. Let K be a component of $S - p$ that contains at least one of the W_i's. It follows from the compactness of the W_i's and Theorem 3 that there exists a continuum $W \subset K$ such that $(S - V) \cap K \subset W$. Since $p \notin T(W)$ there exists a continuum M such that $p \in \text{Int}(M)$ and $M \cap W = \varnothing$. Hence $(M \cap K) \cup \{p\} = M \cap \overline{K}$ is a continuum lying in $V \cap \overline{K}$ containing p in its interior relative to $\overline{K}$. Since there are only a finite number of components of $S - p$ which contain at least one of the W_i's the closure H of their union is connected im kleinen at p. Furthermore $(S - H) \cup \{p\} = N$ is a continuum lying in V. If L denotes the component of $V \cap H$ which contains p then $N \cup L$ is the component of V containing p and $p \in \text{Int}(L)$. Thus S is connected im kleinen at p.

The above proof is patterned after Jones' proof of Theorem 8 of [2].[2]

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COROLLARY 4.1. *Let p be a point of S. If, for every subcontinuum W that does not contain p, S is decomposable about p and W then S is connected im kleinen at p.*

COROLLARY 4.2. *Let p be a point of S. If S is not connected im kleinen at p but is semilocally connected at p then there exists a subcontinuum W of S such that $p \in T(W) - W$.*

COROLLARY 4.3. *Let S be semilocally connected at p. S is connected im kleinen at p iff for all subcontinua W of S, $p \in W$ iff $p \in T(W)$.*

COROLLARY 4.4. *S is locally connected iff for every subcontinuum W of S, $T(W) = W$.*

EXAMPLE 1. The hypothesis in Theorem 4 that $T(p) = p$ is necessary. There exists a subcontinuum S of the plane and a point $p \in S$ such that for no subcontinua W of S is it true that $p \in T(W) - W$ and yet S is not connected im kleinen at p.

Construct S as follows:

Let $p = (0, 0)$, $a = (1, 0)$, $b(-1, 0)$, $a_m = (1, 1/m)$, $b_m = (-1, 1/m)$, $c_m = (1, -1/m)$ and $d_m = (-1, -1/m)$. Let

$$K_m = \left\{(x, y) \mid (x - 1)^2 + y^2 = \left(\frac{1}{m}\right)^2, \ x \geqq 1\right\}$$

and

$$L_m = \left\{(x, y) \mid (x + 1)^2 + y^2 = \left(\frac{1}{m}\right)^2, \ x \leqq -1\right\}.$$

If h and k are points of the plane then denote by $[hk]$ the closed line segment from h to k. Let

$$S = [ab] \cup \left(\bigcup_{m=1}^{\infty} (K_m \cup L_m \cup [pa_m] \cup [pb_m] \cup [c_m d_m])\right).$$

THEOREM 5. *Let p be a point of S. If there exists a collection $\{W_\alpha\}$ of subcontinua of S such that $\{W_\alpha\}$ is simply ordered by inclusion, $S - p = \bigcup\{W_\alpha\} = \bigcup\{\mathrm{Int}(W_\alpha)\}$ and for each α, $p \notin T(W_\alpha)$ then S is connected im kleinen at p.*

PROOF. It is immediate that $T(p) = p$. Let W be a subcontinuum of S that does not contain p. Since W is compact and $\{\mathrm{Int}(W_\alpha)\}$ is simply ordered by inclusion, there is a continuum $W_\beta \in \{W_\alpha\}$ such that $W \subset \mathrm{Int}(W_\beta)$. Since $p \notin T(W_\beta)$, $p \notin T(W)$. Thus, by Theorem 4, S is connected im kleinen at p.

3. *T*-**addition continua.** S is *weakly irreducible* iff the number of components of the complement of any finite collection of subcontinua is finite. S is *T-symmetric* iff for any two closed subsets A and B of S, $T(A)\cap B=\emptyset$ iff $A\cap T(B)=\emptyset$. S is *T-additive* iff for any collection of closed subsets $\{A_\alpha\}$ of S whose union is closed, $T(\bigcup\{A_\alpha\})=\bigcup\{T(A_\alpha)\}$.

THEOREM 6. *If S is weakly irreducible then S is T-symmetric.*

PROOF. Let A and B be closed subsets of S and suppose $A\cap T(B)=\emptyset$. Since A is compact and does not intersect $T(B)$, there is a finite collection $\{W_i\}$ of disjoint subcontinua of S whose interiors cover A, and whose union does not intersect B. Let $p\in B$ and let K be the component of $S-\bigcup\{W_i\}$ that contains p. Since S is weakly irreducible K is open and since $A\subset\{\text{Int}(W_i)\}$, $\overline{K}\cap A=\emptyset$. Hence $p\notin T(A)$ and $B\cap T(A)=\emptyset$. By symmetric argument, if $B\cap T(A)=\emptyset$ then $A\cap T(B)=\emptyset$ and thus S is T-symmetric.

THEOREM 7. *If S is T-symmetric then S is T-additive.*

PROOF. Let $\{A_\alpha\}$ be a collection of closed subsets of S whose union is closed. Since always $T(\bigcup\{A_\alpha\})\supset\bigcup\{T(A_\alpha)\}$, it need only be shown that $T(\bigcup\{A_\alpha\})\subset\bigcup\{T(A_\alpha)\}$. Let $p\in T(\bigcup\{A_\alpha\})$. Since S is T-symmetric $T(p)\cap\bigcup\{A_\alpha\}=\emptyset$; hence there exists β such that $T(p)\cap A_\beta\neq\emptyset$. But then $p\in T(A_\beta)$; hence $p\in\bigcup\{T(A_\alpha)\}$ and thus S is T-additive.

THEOREM 8. *If, for any finite collection $\{W_i\}$ of subcontinua of S and for any point p such that $p\in\bigcap\{\text{Int}(W_i)\}$, there exists a continuum W such that $p\in\text{Int}(W)$ and $W\subset\bigcap\{W_i\}$ then S is T-additive.*

PROOF. Let $\{A_\alpha\}$ be a collection of closed subsets of S such that $\bigcup\{A_\alpha\}=\text{Cl}(\bigcup\{A_\alpha\})$. Clearly $T(\bigcup\{A_\alpha\})\supset\bigcup\{T(A_\alpha)\}$. Suppose $p\in T(\bigcup\{A_\alpha\})-\bigcup\{T(A_\alpha)\}$. Then, for each α, there exists a subcontinuum W_α of S such that $p\in\text{Int}(W_\alpha)$ and $W_\alpha\cap A_\alpha=\emptyset$. $\{S-W_\alpha\}$ covers $\bigcup\{A_\alpha\}$ which is closed and there forecompact. Let $\{S-W_i\}$ be a finite subcover of $\bigcup\{A_\alpha\}$. But then $p\in\bigcap\{\text{Int}(W_i)\}$. By hypothesis, there exists a subcontinuum W of S such that $p\in\text{Int}(W)$ and $W\subset\bigcap\{W_i\}$. Hence $p\notin T(\bigcup\{A_\alpha\})$ contradicting the supposition. Thus $T(\bigcup\{A_\alpha\})=\bigcup\{T(A_\alpha)\}$.

COROLLARY 8.1. *Let S be compact. If S is hereditarily unicoherent then S is T-additive.*

THEOREM 9. *Let S be T-additive and $p\in S$. If S is not connected* *im kleinen at p then there exists a subcontinuum W of S such that $p \in T(W) - W$.*

PROOF. Since S is not connected im kleinen at p, there exists an open set Q containing p such that $p \in T(\mathrm{Fr}(Q))$. Let $\{W_\alpha\}$ be the components of $\mathrm{Fr}(Q)$ then $\mathrm{Fr}(Q) = \bigcup\{W_\alpha\} = \mathrm{Cl}(\bigcup\{W_\alpha\})$ and the W_α's are closed. Thus $p \in \bigcup T(W_\alpha)$; so there exists $W \in \{W_\alpha\}$ such that $p \in T(W)$.

COROLLARY 9.1. *Let S be T-additive and $p \in S$. S is connected im kleinen at p iff, for all subcontinua W of S $p \in W$ iff $p \in T(W)$.*

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