DENSITIES IN ARITHMETIC PROGRESSIONS

MORRIS GOLDFELD

Let $S=\{s_1, s_2, \cdots\}$ be a set of positive integers. Then the density of S (denoted by $d(S)$) is the $\lim_{n\to\infty} S(n)/n$, if the limit exists, where $S(n)$ is the number of integers in S that are less than or equal to n. Clearly, if A is an arithmetic progression of difference a, then $d(A)=1/a$.

If we consider the algebra consisting of all finite unions of arithmetic progressions, then it can easily be shown that the density function is a finitely additive measure on this algebra. The chief obstruction to our knowledge about the density function lies in the fact that the density does not extend to the σ-algebra. In certain cases, however, it does. This paper is concerned with those cases; and in particular with the arithmetic progressions A_i with differences a_i satisfying the following condition

$$d(\cap \bar{A}_i) = \prod (1 - 1/a_i) \tag{1}$$

where the intersection and product run through $i=1, 2, 3, \cdots$ and where $\bar{A}_i$ denotes the complement of A_i. It will be shown if the preceding condition is satisfied that one can give fairly simple expressions for the density of $\cap \bar{A}_i$ in an arithmetic progression, if the density exists in that progression. It seems to be true that if (1) holds then $d((\cap \bar{A}_i)\cap B)$ exists for any arithmetic progression B although I do not see how to prove it.

Let $\{a_1, a_2, \cdots\}$ be a set of pairwise relatively prime positive integers. Let A_i be the set of all positive multiples of a_i. Put $S=\cap \bar{A}_i$. We shall prove the following general theorem on the density of $S\cap B$, where B is the set of all positive multiples of b for an arbitrary positive integer b.

THEOREM. *Let $b=b'q_1q_2 \cdots q_s$ where b' is (for each i) relatively prime to (a_i/q_i) and q_k is the greatest common divisor of b and a_k. Then if*

$$d(S) = \prod (1 - 1/a_i),$$

we must also have

$$d(B \cap S) = \frac{1}{b'} \prod_{k=1}^{s} \left(\frac{a_k/q_k - 1}{a_k - 1}\right) \prod (1 - 1/a_i)$$

if the density of $B\cap S$ exists.

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PROOF. Let $|A|$ denote the number of elements in any set A. Also let $I_n = \{1, 2, \cdots n\}$ be the set containing the first n integers.

Put $S_r = \bigcap_{i=1}^r \overline{A}_i$ and $S_r(n) = |S_r \cap I_n|$. We consequently have by virtue of the definition of density

$$d(S) = \lim_{n\to\infty} \frac{1}{n}\left(\lim_{r\to\infty} S_r(n)\right)$$

Since $d(S)$ exists (by assumption) it does not matter through what sequence of n we reach our limit, as long as we choose n constantly increasing.

Let $n_j = \prod_{i=1}^j a_i$. We then have,

$$d(S) = \lim_{j\to\infty} \frac{1}{bn_j}\left(\lim_{r\to\infty} S_r(bn_j)\right).$$

But $\overline{A}_v \cap I_{bn_j} = I_{bn_j}$ for $v \geqq bn_j$ since $\overline{A}_i$ contains all the integers less than a_i for each i. Hence,

$$\lim_{r\to\infty} S_r(bn_j) = S_{bn_j}(bn_j)$$
$$= S_j(bn_j) - X(b, j),$$

where $X(b, j)$ is the error.

On making use of the exclusion-inclusion principle, we see that

$$S_j(bn_j) = b\prod_{i=1}^{j}(a_i - 1).$$

Putting all this back into the expression for density yields

$$d(S) = \prod (1 - 1/a_i) - \lim_{j\to\infty} X(b, j)/n_j,$$

and since $d(S) = \prod(1 - 1/a_i)$ by assumption, this makes

$$\lim_{j\to\infty} X(b, j)/n_j = 0. \tag{2}$$

Now, let $S_r(B, n) = |S_r \cap B \cap I_n|$. Then

$$d(B \cap S) = \lim_{j\to\infty} \frac{1}{bn_j}\left(\lim_{r\to\infty} S_r(B, bn_j)\right)$$
$$= \lim_{j\to\infty} \frac{1}{bn_j}\left(S_{bn_j}(B, bn_j)\right)$$

as before. We now put

$$S_{bn_j}(B, bn_j) = S_j(B, bn_j) - X^1(b, j)$$

where $X^1(b, j) \leqq X(b, j)$ for all j. Making use of the exclusion-inclusion principle once again, we see that

$$S_j(B, bn_j) = \frac{b}{b'} \prod_{k=1}^{s} \frac{a_k/q_k - 1}{a_k - 1} \prod_{i=1}^{j} (a_i - 1).$$

Therefore,

$$d(B \cap S) = \frac{1}{b'} \prod_{k=1}^{s} \frac{a_k/q_k - 1}{a_k - 1} \prod (1 - 1/a_i) - \lim_{j \to \infty} X(b, j)/n_j,$$

and since $X^1(b, j) \leqq X(b, j)$, this makes (by equation (2))

$$\lim_{j \to \infty} X^1(b, j)/n_j = 0. \qquad \text{Q.E.D.}$$

COROLLARY (1). *Let $\{a_1, a_2, \cdots\}$ be a set of pairwise relatively positive integers with A_i denoting the set of positive multiples of a_i. Also, let X_k denote the number of integers less than $\prod_1^k a_i$ which are not divisible by any a_i, $0 < i \leqq k$, and divisible by some a_j where $k < j \leqq \prod_1^k a_i$. Then if* (1) *holds, we must also have*

$$\lim_{k \to \infty} \frac{X_k}{\prod_1^k a_i} = 0.$$

PROOF. The proof follows immediately from the proof of the preceding theorem (by equation (2)) if we can show that

$$X_k = S_k(n_k) - S_{n_k}(n_k) = X(1, k). \tag{3}$$

But the right side of (3) is just what we mean by X_k. Q.E.D.

As an application of the theorem we consider the problem of finding the density of squarefree integers in an arithmetic progression.

COROLLARY (2). *Let $B = \{b, 2b, 3b, \cdots\}$ where b is squarefree and can be factored into primes $q_1 q_2 \cdots q_s$; then we have (for S = squarefree integers)*

$$d(B \cap S) = \frac{6}{\pi^2} \prod_{i=1}^{s} (1/(q_i + 1))$$

if the density exists.

PROOF. Let A_i consist of the positive multiples of q_i^2 for $i = 1, \cdots, s$. Let $\{p_{s+1}, p_{s+2}, \cdots\}$ be the set of all primes relatively prime to b.

We let A_j denote the set of all positive multiples of p_j^2 for $j>s$. Then $S=\bigcap \overline{A}_j$ is nothing more than the set of squarefree integers. But we already know that [**1**, p. 269]

$$d(S) = \prod (1 - 1/p^2) = 6/\pi^2,$$

where the product is over all primes p. Hence S satisfies the condition (1). By the result of our theorem, we must therefore have

$$\begin{aligned} d(B \cap S) &= \prod_{k=1}^{s} \frac{q_k^2/q_k - 1}{q_k^2 - 1} \prod (1 - 1/p^2) \\ &= \frac{6}{\pi^2} \prod 1/(q_k + 1). \end{aligned}$$

Q.E.D.

Reference

1. G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

Columbia University