DENSITIES IN ARITHMETIC PROGRESSIONS

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Let $S = \{ s_1, s_2, \cdots \}$ be a set of positive integers. Then the density of $S$ (denoted by $d(S)$) is the \( \lim_{n \to \infty} S(n)/n \), if the limit exists, where $S(n)$ is the number of integers in $S$ that are less than or equal to $n$. Clearly, if $A$ is an arithmetic progression of difference $a$, then $d(A) = 1/a$.

If we consider the algebra consisting of all finite unions of arithmetic progressions, then it can easily be shown that the density function is a finitely additive measure on this algebra. The chief obstruction to our knowledge about the density function lies in the fact that the density does not extend to the $\sigma$-algebra. In certain cases, however, it does. This paper is concerned with those cases; and in particular with the arithmetic progressions $A_i$ with differences $a_i$ satisfying the following condition

\[
d(\bigcap A_i) = \prod (1 - 1/a_i)
\]

where the intersection and product run through $i = 1, 2, 3, \cdots$ and where $\overline{A}_i$ denotes the complement of $A_i$. It will be shown if the preceding condition is satisfied that one can give fairly simple expressions for the density of $\bigcap A_i$ in an arithmetic progression, if the density exists in that progression. It seems to be true that if (1) holds then $d((\bigcap A_i) \cap B)$ exists for any arithmetic progression $B$ although I do not see how to prove it.

Let $\{a_1, a_2, \cdots \}$ be a set of pairwise relatively prime positive integers. Let $A_i$ be the set of all positive multiples of $a_i$. Put $S = \bigcap A_i$. We shall prove the following general theorem on the density of $S \cap B$, where $B$ is the set of all positive multiples of $b$ for an arbitrary positive integer $b$.

**Theorem.** Let $b = b'q_1q_2 \cdots q_s$, where $b'$ is (for each $i$) relatively prime to $(a_i/q_i)$ and $q_k$ is the greatest common divisor of $b$ and $a_k$. Then if

\[d(S) = \prod (1 - 1/a_i),\]

we must also have

\[d(B \cap S) = \frac{1}{b'} \prod_{k=1}^{s} \left( \frac{a_k/q_k - 1}{a_k - 1} \right) \prod (1 - 1/a_i)\]

if the density of $B \cap S$ exists.

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Proof. Let \( |A| \) denote the number of elements in any set \( A \). Also let \( I_n = \{1, 2, \ldots, n\} \) be the set containing the first \( n \) integers.

Put \( S_r = \bigcap_{i=1}^{\infty} A_i \) and \( S_r(n) = |S_r \cap I_n| \). We consequently have by virtue of the definition of density

\[
d(S) = \lim_{n \to \infty} \frac{1}{n} \left( \lim_{r \to \infty} S_r(n) \right)
\]

Since \( d(S) \) exists (by assumption) it does not matter through what sequence of \( n \) we reach our limit, as long as we choose \( n \) constantly increasing.

Let \( n_j = \prod_{i=1}^{l-1} a_i \). We then have,

\[
d(S) = \lim_{j \to \infty} \frac{1}{b_{nj}} \left( \lim_{r \to \infty} S_r(b_{nj}) \right).
\]

But \( A_v \cap I_{b_{nj}} = I_{b_{nj}} \) for \( v \geq b_{nj} \) since \( A_i \) contains all the integers less than \( a_i \) for each \( i \). Hence,

\[
\lim_{r \to \infty} S_r(b_{nj}) = S_{b_{nj}}(b_{nj}) = S_{b_{nj}}(b_{nj}) - X(b, j),
\]

where \( X(b, j) \) is the error.

On making use of the exclusion-inclusion principle, we see that

\[
S_{b_{nj}}(b_{nj}) = b \prod_{i=1}^{j} (a_i - 1).
\]

Putting all this back into the expression for density yields

\[
d(S) = \prod_{i=1}^{j} (1 - 1/a_i) - \lim_{j \to \infty} X(b, j)/n_j,
\]

and since \( d(S) = \prod_{i=1}^{j}(1-1/a_i) \) by assumption, this makes

\[
(2) \quad \lim_{j \to \infty} X(b, j)/n_j = 0.
\]

Now, let \( S_r(B, n) = |S_r \cap B \cap I_n| \). Then

\[
d(B \cap S) = \lim_{j \to \infty} \frac{1}{b_{nj}} \left( \lim_{r \to \infty} S_r(B, b_{nj}) \right) = \lim_{j \to \infty} \frac{1}{b_{nj}} (S_{b_{nj}}(B, b_{nj}))
\]

as before. We now put
\[ S_{bn_j}(B, bn_j) = S_j(B, bn_j) - X^1(b, j) \]

where \( X^1(b, j) \leq X(b, j) \) for all \( j \). Making use of the exclusion-inclusion principle once again, we see that

\[ S_j(B, bn_j) = \frac{b}{b'} \prod_{k=1}^{s} \frac{a_k/q_k - 1}{a_k - 1} \prod_{i=1}^{j} (a_i - 1). \]

Therefore,

\[ d(B \cap S) = \frac{1}{b'} \prod_{k=1}^{s} \frac{a_k/q_k - 1}{a_k - 1} \prod_{i=1}^{j} (1 - 1/a_i) - \lim_{j \to \infty} X(b, j)/n_j, \]

and since \( X^1(b, j) \leq X(b, j) \), this makes (by equation (2))

\[ \lim_{j \to \infty} X^1(b, j)/n_j = 0. \]

**Q.E.D.**

**Corollary (1).** Let \( \{a_1, a_2, \ldots\} \) be a set of pairwise relatively positive integers with \( A_i \) denoting the set of positive multiples of \( a_i \). Also, let \( X_k \) denote the number of integers less than \( \prod_i a_i \) which are not divisible by any \( a_i \), \( 0 < i \leq k \), and divisible by some \( a_j \) where \( k < j \leq \prod_i a_i \). Then if (1) holds, we must also have

\[ \lim_{k \to \infty} \frac{X_k}{\prod_i a_i} = 0. \]

**Proof.** The proof follows immediately from the proof of the preceding theorem (by equation (2)) if we can show that

(3)  
\[ X_k = S_k(n_k) - S_{nk}(n_k) = X(1, k). \]

But the right side of (3) is just what we mean by \( X_k \).  

**Q.E.D.**

As an application of the theorem we consider the problem of finding the density of squarefree integers in an arithmetic progression.

**Corollary (2).** Let \( B = \{b, 2b, 3b, \ldots\} \) where \( b \) is squarefree and can be factored into primes \( q_1 q_2 \cdots q_s \); then we have (for \( S = \) squarefree integers)

\[ d(B \cap S) = \frac{6}{\pi^2} \prod_{i=1}^{s} (1/(q_i + 1)) \]

if the density exists.

**Proof.** Let \( A_i \) consist of the positive multiples of \( q_i^2 \) for \( i = 1, \ldots, s \). Let \( \{p_{s+1}, p_{s+2}, \ldots\} \) be the set of all primes relatively prime to \( b \).
We let $A_j$ denote the set of all positive multiples of $p_j^2$ for $j > s$. Then $S = \bigcap A_j$ is nothing more than the set of squarefree integers. But we already know that [1, p. 269]

$$d(S) = \prod (1 - 1/p^2) = 6/\pi^2,$$

where the product is over all primes $p$. Hence $S$ satisfies the condition (1). By the result of our theorem, we must therefore have

$$d(B \cap S) = \prod_{k=1}^{s} \frac{q_k^2/q_k - 1}{q_k^2 - 1} \prod (1 - 1/p^2)$$

$$= \frac{6}{\pi^2} \prod 1/(q_k + 1).$$

Q.E.D.

Reference


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