BOUNDED FUNCTIONS WITH LARGE CIRCULAR VARIATION

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In 1951, S. N. Mergeljan [1] proved that *there exists a bounded holomorphic function f for which*

$$\iint_{|z|<1} |f'(z)|\, dS = \infty. \tag{1}$$

An obvious geometric interpretation of (1) is that the length $l(r)$ of the image of the circle $|z| = r$ grows so rapidly, as $r \to 1$, that $l(r)$ is not an integrable function of r.

An alternate geometric interpretation of (1) is that the length $V(f, \theta)$ of the image of the radius of $e^{i\theta}$ is not an integrable function of θ. W. Rudin [2, Theorem III] has proved a proposition stronger than Mergeljan's, namely, that there exist Blaschke products $B(z)$ such that $V(B, \theta) = \infty$ for almost all θ. It follows that there exists a function f, holomorphic in the unit disk D and continuous in the closure of D, such that $V(f, \theta) = \infty$ for almost all θ [2, Theorem IV].

Both Mergeljan's and Rudin's arguments involve nonconstructive steps, and therefore they do not allow us to visualize the functions f in terms of any of the customary representations. In this note, I give two explicit constructions that prove Mergeljan's result. Unfortunately, my examples are inadequate for Rudin's theorem.

We begin with the function $(a^n - z^n)/(1 - a^n z^n)$, where $2^{-1/n} < a < 1$. We write $a^n = \alpha$ and $z^n = \zeta$, and we observe that for $0 < \rho < \alpha$, the maximum and minimum values of $|(\alpha - \zeta)/(1 - \alpha\zeta)|$ on the circle $|\zeta| = \rho$ are

$$(\alpha + \rho)/(1 + \alpha\rho) \quad \text{and} \quad (\alpha - \rho)/(1 - \alpha\rho),$$

respectively. The difference between the two moduli is $2\rho(1 - \alpha^2)/(1 - \alpha^2\rho^2)$. Therefore the function $(a^n - z^n)/(1 - a^n z^n)$, whose $2n$ points of maximum and minimum modulus on the circle $|z| = r$ separate each other, maps that circle onto a curve of length greater than

$$2n \cdot 2r^n(1 - a^{2n})/(1 - a^{2n}r^{2n}) \qquad (0 < r < a).$$

The integral of this quantity, taken over the interval $3^{-1/n} < r < a$,

Received by the editors August 27, 1967.

[1] This work was supported by the National Science Foundation.

is greater than $K_1 n(1-a)|\log n(1-a)|$, where K_1 is a constant independent of a and n.

We now consider the Blaschke product

$$B(z) = \prod \frac{a_k^{n_k} - z^{n_k}}{1 - a_k^{n_k} z^{n_k}}.$$

The product converges if $\sum n_k(1-a_k) < \infty$, in particular, if

$$n_k(1 - a_k) = 1/k(\log k)^{3/2} \qquad (k = 2, 3, \cdots).$$

If moreover the sequence $\{n_k\}$ increases fast enough, we obtain disjoint intervals $r_k < r < a_k$ such that

$$\int_{r_k}^{a_k}\int_0^{2\pi} |B'(re^{i\theta})| r d\theta dr > K_2 n_k(1 - a_k)|\log n_k(1 - a_k)|$$
$$> K_2/k(\log k)^{1/2},$$

and Mergeljan's theorem is proved.

From our construction, we see immediately that *there exists a continuous function f satisfying condition* (1). Indeed, it is sufficient to choose finite Blaschke products B_m such that, for each of certain disjoint concentric annuli A_m,

$$\iint_{A_m} |B'_m| dS - \sum_{j\neq m}\iint_{A_j} |B'_j| dS > m^3,$$

and to take $f(z) = \sum m^{-2}B_m(z)$.

Our second example is based on the function

$$g(z) = \exp\left(-a\frac{1 + z^n}{1 - z^n}\right).$$

Since the maximum and minimum modulus of $g(z)$ on the circle $|z^n| = \rho$ are

$$\exp\left(-a\frac{1 - \rho}{1 + \rho}\right) \quad \text{and} \quad \exp\left(-a\frac{1 + \rho}{1 - \rho}\right),$$

the function g maps the circle C_r onto a curve of length greater than

$$2n\left\{\exp\left(-a\frac{1 - r^n}{1 + r^n}\right) - \exp\left(-a\frac{1 + r^n}{1 - r^n}\right)\right\}.$$

To estimate the integral of this lower bound over the interval $0 < r < 1$, we make the substitution $s=(1-r^n)/(1+r^n)$, and of the resulting integral

$$4\int_0^1 (e^{-as}-e^{-a/s})(1-s)^{-1+1/n}(1+s)^{-1-1/n}ds$$

we discard everything except the portion over $(0, a^{1/2})$. We may then replace the algebraic factors by a constant, and the quantity to be determined is greater than

$$K_3\int_0^{a^{1/2}} (e^{-a}-e^{-a/s})ds.$$

Consider separately each of the intervals $[(j-1)a, ja]$ $(j=1, 2, \cdots, [a^{-1/2}])$. Since the minimum of the integrand in the jth interval is

$$e^{-a}-e^{-1/j} > -a+j^{-1}-j^{-2},$$

the value of the integral is greater than

$$a\sum_{j=1}^{[a^{-1/2}]} [-a+j^{-1}-j^{-2}] > K_4 a\,|\log a|\,.$$

Now, for $k=2, 3, \cdots$, let $a_k=k^{-1}(\log k)^{-3/2}$, and let

$$f(z) = \exp\left(-\sum a_k \frac{1+z^{n_k}}{1-z^{n_k}}\right).$$

If $n_k\to\infty$ fast enough, then f again has the desired properties.

REFERENCES

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