ON THE ZEROS OF VAN VLECK POLYNOMIALS

G. M. SHAH[1]

1. **Introduction.** It is known [4] that there exist $C_{n+p-2,p-2}$ polynomials $V(x)$ of degree $(p-2)$ such that corresponding to each such $V(x)$ the differential equation

$$\prod_{j=1}^{p} (x-a_j)\left[y'' + \left(\sum_{j=1}^{p} \frac{\alpha_j}{x-a_j}\right)y'\right] + V(x)y = 0, \tag{1.1}$$

where all $\alpha_j > 0$ and $a_1 < a_2 < \cdots < a_p$, has a unique polynomial solution $S(x)$ of degree n. Such $S(x)$ and $V(x)$ are called *Stieltjes* and *Van Vleck* polynomials respectively [2]. It has been shown that the zeros of all such $S(x)$ and $V(x)$ lie in (a_1, a_p) [1] and [5]. We have proved that each $V(x)$ can have at most two zeros in any interval (a_r, a_{r+1}), $2 \leqq r \leqq p-2$ and at most one zero in each of the intervals (a_1, a_2) and (a_{p-1}, a_p) [3b]. We can, however, improve this result if we consider the distribution of the zeros of $V(x)$ in k $(k>1)$ consecutive intervals. We intend to prove the following result:

THEOREM 1. (a) *Any k $(k \leqq p-1)$ consecutive intervals $(a_j, a_{j+1}), \cdots, (a_{j+k-1}, a_{j+k})$ contain at most k and at least $(k-1)$ zeros of $V(x)$, if $j=1$ or $j=p-k$.*

(b) *Any k $(k<p-1)$ consecutive intervals $(a_j, a_{j+1}), \cdots, (a_{j+k-1}, a_{j+k})$ contain at most $(k+1)$ and at least $(k-1)$ zeros of $V(x)$, if $j \neq 1$ and $j \neq p-k$.*

In §2 we prove a result analogous to Rolle's theorem for the zeros of the derivative of a function with real and simple zeros. In §3 we give the proof of Theorem 1.

2. **Analogue of Rolle's theorem.** As pointed out in an earlier paper [3b] we may assume that no $V(x)$ has a zero at any a_j, $1 \leqq j \leqq p$, for the modification in the proof in the other case will be the obvious one. We recall from [3b] the following

LEMMA 1. *The zeros of a Van Vleck polynomial $V(x)$ of degree $(p-2)$ and those of the corresponding $S(x)$ of degree n are the zeros of the function*

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(2.1) $$G(x) \equiv \sum_{j=1}^{n-1} \frac{1}{x - x_j'} + \sum_{j=1}^{p} \frac{\alpha_j}{x - a_j},$$

and conversely, where x_j' $(1 \leqq j \leqq n-1)$ *are the zeros of the derivative* $S'(x)$ *of the* $S(x)$.

Let E denote the point set consisting of the zeros of $S(x)$ and those of the corresponding $V(x)$. Also, let F consist of the singular points $a_1, a_2, \cdots, a_p$ and the zeros of the derivative $S'(x)$ of the $S(x)$. Then, we have the following analogue of Rolle's theorem for the sets E and F.

THEOREM 2. *Between any two consecutive points of E lies one and only one point of F.*

PROOF. We prove first that between two consecutive points of E lies one point of F. We consider various cases separately.

Case 1. Let the two points of E under consideration be x_k and x_{k+1}, both zeros of $S(x)$. Since the zeros of $S(x)$ are real and simple [**3a**], it follows that one zero x_k' of $S'(x)$ lies between x_k and x_{k+1}. x_k' is a point of F and we have $x_k < x_k' < x_{k+1}$.

Case 2. Let the two points of E be x_k, a zero of $S(x)$ and t_l, a zero of the corresponding $V(x)$. Then, by Lemma 1, x_k and t_l are both zeros of $G(x)$. It is easy to see that $G(x)$ has points of discontinuity at the points a_j $(1 \leqq j \leqq p)$ and at x_j' $(1 \leqq j \leqq n-1)$. Also $G(x)$ is a continuously decreasing function of x in every interval of continuity. As $G(x_k) = 0$ and $G(t_l) = 0$, it follows that $G(x)$ must have a discontinuity between x_k and t_l, for otherwise it would contradict the monotonic character of $G(x)$. That point of discontinuity of G, being either an a_j or a x_j', is a point of F.

Case 3. Let the two points of E be t_k and t_{k+1}, both zeros of $V(x)$. It can be shown as in Case 2 above, that the interval (t_k, t_{k+1}) contains a point of discontinuity of G which is a point of F.

We show that between any two consecutive points of F is a point of E.

Case a. Suppose the two consecutive points of F are a_k, a_{k+1}. Then $G(x)$ decreases continuously from $+\infty$ to $-\infty$ as x moves from a_k to a_{k+1}, because the interval (a_k, a_{k+1}) does not contain any x_j'. Thus G vanishes precisely once in (a_k, a_{k+1}). Since E consists of the zeros of G, the result follows in this case.

Case b. Let the two consecutive points of F be a_k, a_r', a zero of $S'(x)$. For convenience, suppose that $a_k < x_r'$. (In case $a_k = x_r'$, then it can be shown that $V(x)$ has also a zero at this a_k [**3b**].) Then again,

$G(x)$ decreases continuously from $+\infty$ to $-\infty$ as x varies from a_k to x_r'. Thus the interval (a_k, x_r') contains a point of E, namely the zero of G in (a_k, x_r').

Case c. Suppose the two consecutive points of F are x_k' and x_{k+1}', both zeros of $S'(x)$. Then, since the zeros of $S(x)$ are all real and simple [3a], $S(x)$ has one zero in (x_k', x_{k+1}') and this completes the proof.

It may be noted that an a_j and a zero x_i' of $S'(x)$ may coincide. We have shown that this is possible if and only if this a_j is also a zero of $V(x)$ [**3b**]. We show now that no two points of E can coincide. It is known that zeros of $S(x)$ are simple and those of $V(x)$ are also distinct. It then suffices to prove the following

THEOREM 3. *No zero of a Van Vleck polynomial $V(x)$ is a zero of the corresponding Stieltjes polynomial $S(x)$.*

PROOF. If possible, let x_k be a zero of both the polynomials $V(x)$ and $S(x)$. Let $S(x)=(x-x_k)T(x)$, then $T(x_k)\neq 0$ and $T(x)$ has all its zeros distinct. Also,

$$S'(x) = (x - x_k)T'(x) + T(x), \qquad S''(x) = (x - x_k)T''(x) + 2T'(x),$$
$$S'''(x) = (x - x_k)T'''(x) + 3T''(x).$$

Thus,

$$S'(x_k) = T(x_k) \neq 0, \qquad S''(x_k) = 2T'(x_k), \qquad S'''(x_k) = 3T''(x_k).$$

Let $V(x)=(x-x_k)R(x)$. Then, equation (1.1) becomes,

$$S''(x) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{x - a_j}\right)S'(x) = -\frac{(x - x_k)^2T(x)R(x)}{\prod_{j=1}^{p}(x - a_j)}. \tag{2.2}$$

Equation (2.2) shows that the function

$$F(x) \equiv S''(x) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{x - a_j}\right)S'(x)$$

has a double zero at x_k. Thus, $F(x_k)=F'(x_k)=0$. Let

$$T(x) = \prod_{j=1;\, j\neq k}^{n} (x - x_j),$$

then x_j are real, distinct and $x_j\neq x_k$, $(j=1, \cdots, n;\, j\neq k)$. Hence,

$$\frac{T'(x_k)}{T(x_k)} = \sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j} \tag{2.3}$$

and

$$\text{(2.4)}\qquad \begin{aligned} \frac{T''(x_k)}{T(x_k)} &= \sum_{r<j=1;\, j,r\neq k}^{n} \frac{2}{(x_k - x_j)(x_k - x_r)} \\ &= \left[\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j}\right]^2 - \sum_{j=1;\, j\neq k}^{n} \frac{1}{(x_k - x_j)^2}. \end{aligned}$$

Also,

$$\frac{S''(x_k)}{S'(x_k)} = \frac{2T'(x_k)}{T(x_k)} = 2\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j}.$$

Now, $F(x_k) = 0$ gives

$$2\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j} + \sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j} = 0,$$

or,

$$\text{(2.5)}\qquad 2\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j} = -\sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j}.$$

Also, $F'(x_k) = 0$ gives,

$$3T''(x_k) + 2\left(\sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j}\right) T'(x_k) - \left(\sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2}\right) T(x_k) = 0,$$

or, since $T(x_k) \neq 0$,

$$3\,\frac{T''(x_k)}{T(x_k)} + 2\left(\sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j}\right)\frac{T'(x_k)}{T(x_k)} - \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0.$$

Using equations (2.3) and (2.4), this reduces to

$$\begin{aligned} 3\left[\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j}\right]^2 &- 3\sum_{j=1;\, j\neq k}^{n} \frac{1}{(x_k - x_j)^2} \\ &+ 2\left(\sum_{j=1}^{p} \frac{\alpha_j}{x_k - a_j}\right)\left(\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j}\right) \\ &- \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0. \end{aligned}$$

In view of equation (2.5), this can be written as

$$\text{(2.6)}\qquad \begin{aligned} -3\sum_{j=1;\, j\neq k}^{n} \frac{1}{(x_k - x_j)^2} &- \left[\sum_{j=1;\, j\neq k}^{n} \frac{1}{x_k - x_j}\right]^2 \\ &- \sum_{j=1}^{p} \frac{\alpha_j}{(x_k - a_j)^2} = 0. \end{aligned}$$

In the left-hand side of equation (2.6), each term is negative and hence the sum cannot vanish. This contradiction leads to the desired result.

3. **Proof of Theorem 1.** Let the k consecutive intervals (a_j, a_{j+1}), $\cdots$, (a_{j+k-1}, a_{j+k}) contain r $(1 \leqq r \leqq n)$ zeros of $S(x)$. Let these zeros be $x_{m+1} < \cdots < x_{m+r}$. Then the following situations arise:

Case I. The intervals (a_j, x_{m+1}) and (x_{m+r}, a_{j+k}) contain each one zero of $S'(x)$. In this case the interval (a_j, a_{j+k}) contains precisely $(r+1)$ zeros of $S'(x)$. This interval, therefore, contains $(k+r+2)$ points of the set F. By Theorem 2, therefore, there are exactly $(k+r+1)$ points of E in these k intervals. Since we have supposed that these intervals contain r zeros of $S(x)$, it follows that the intervals (a_j, a_{j+1}), $\cdots$, (a_{j+k-1}, a_{j+k}) contain exactly $(k+1)$ zeros of $V(x)$.

Case II. The interval (a_j, x_{m+1}) contains one zero of $S'(x)$ and (x_{m+r}, a_{j+k}) does not contain any zero of $S'(x)$. In such a case, $S'(x)$ has r zeros in (a_j, a_{j+k}) and hence (a_j, a_{j+k}) contains $(k+r+1)$ points of F. By Theorem 2, there are exactly $(k+r)$ points of E. Thus (a_j, a_{j+k}) contains k zeros of $V(x)$.

Case III. Neither of the intervals (a_j, x_{m+1}) and (x_{m+r}, a_{j+k}) contains any zero of $S'(x)$. Here (a_j, a_{j+k}) contains $(r-1)$ zeros of $S'(x)$ and hence $(k+r)$ points of F. By Theorem 2, (a_j, a_{j+k}) contains exactly $(k+r-1)$ points of E and hence $(k-1)$ zeros of $V(x)$.

In case $j=1$, the situation of Case I cannot prevail. For, in this case, (a_1, x_m) does not contain any zero of $S'(x)$, because x_m is the smallest zero of $S(x)$ in (a_1, a_{k+1}). Also, if $j=p-k$, the Case I is not possible, because (x_m, a_p) does not contain any zero of $S'(x)$. In this case, x_m is the largest zero of $S(x)$.

Finally we consider the case when the k intervals (a_j, a_{j+1}), $\cdots$, (a_{j+k-1}, a_{j+k}) do not contain any zero of $S(x)$. In such a situation the interval (a_j, a_{j+k}) contains at most one zero of $S'(x)$ and hence at most $(k+2)$ points of F. By Theorem 2, therefore, (a_j, a_{j+k}) contains at most $(k+1)$ points of E, or equivalently, at most $(k+1)$ zeros of $V(x)$. It may be noted, however, that in this case $V(x)$ will have at least k zeros in (a_j, a_{j+k}), and that being possible only when $S'(x)$ has no zero in (a_j, a_{j+k}). It is easy to see that when $j=1$ or $j=p-k$, $S'(x)$ has no zero in (a_j, a_{j+k}) and hence $V(x)$ has precisely k zeros in this interval in such a case. The proof of Theorem 1 is now complete.

If we set in Theorem 1, $k=p-1$ and $j=1$ and note that each $V(x)$ is a polynomial of degree $(p-2)$ we have the following result due to Van Vleck [**5**].

THEOREM (VAN VLECK). *All the zeros of* $V(x)$ *lie in* (a_1, a_p).

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UNIVERSITY OF WISCONSIN-MILWAUKEE