PRIME IDEAL STRUCTURE OF RINGS OF BOUNDED CONTINUOUS FUNCTIONS

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Introduction. The order structure of the family of prime ideals in the ring C of all real-valued continuous functions on a topological space has been extensively studied; in this paper we study the analogous problem in the subring C^* of *bounded* functions. The fundamental property of prime ideals in C^* is the following.

MAIN THEOREM. *Let M^* be any maximal ideal of C^* and let M be the unique maximal ideal of C such that the prime ideal $M \cap C^*$ is contained in M^*. Then every prime ideal contained in M^* is comparable with $M \cap C^*$.*

The proof involves topological properties of the Stone-Čech compactification βX of a completely regular Hausdorff space X.

Of special interest are the prime *z-ideals* of C^*. When X is a locally compact, σ-compact Hausdorff space, we show that the family of prime z-ideals of $C^*(X)$ contained in M^* is composed of two subfamilies, order-isomorphic with naturally corresponding families of prime z-ideals in the rings $C(X)$ and $C(\beta X - X)$.

1. **Preliminaries.** We shall use the terminology and notation of the Gillman-Jerison text [**3**]. Applying [**3**, Theorem 3.9], we immediately reduce the problem of the prime ideal structure of $C^*(X)$, and its relation to $C(X)$, to the case that X is a completely regular Hausdorff space. A basic property of prime ideals in rings of functions that will be used several times is a theorem of Kohls ([**9**, Theorem 2.4], see also [**3**, 14.8(a), 6.6(c)]): *In the ring $C(X)$, and also in $C^*(X)$, the prime ideals containing a given prime ideal form a chain.*

The proof of the main theorem is based on Kohls' result and the following theorem ([**10**, 4.4]; cf. [**7**, 3.1] and [**6**, p. 112]): *A prime z-filter $\mathcal{Q}$ on a space T is minimal if and only if for every zero-set Z in $\mathcal{Q}$ there is a zero-set W not in $\mathcal{Q}$ such that $Z \cup W = T$.*

We shall use $\mathfrak{M}^p$ and $\mathcal{O}^p$ to denote the z-filters $\boldsymbol{Z}[M^p]$ and $\boldsymbol{Z}[O^p]$, respectively.

2. **The main theorem.** Under the reduction made in §1 to the case of a completely regular Hausdorff space X, a maximal ideal of $C^*(X)$ corresponds to a point p of βX and is denoted M^{*p}, and the maximal ideal M^p of $C(X)$ that corresponds to p is the unique maximal ideal

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of $C(X)$ such that $M^p \cap C^* \subseteq M^{*p}$ (see [3, Chapter 7]). Thus the main theorem takes the following form.

THEOREM I. *Let $p \in \beta X$. Every prime ideal P of $C^*(X)$ contained in M^{*p} is comparable with $M^p \cap C^*$. Specifically, $P \subseteq M^p \cap C^*$ if and only if P contains no unit of C, while $M^p \cap C^* \subset P$ if and only if P contains a unit of C.*

PROOF. Let P be any prime ideal of $C^*(X)$ with $P \subseteq M^{*p}$. Choose a minimal prime ideal Q with $Q \subseteq P$. To prove that P and $M^p \cap C^*$ are comparable, it suffices to show that $Q \subseteq M^p \cap C^*$, for then P and $M^p \cap C^*$ both contain the prime ideal Q.

To show that $Q \subseteq M^p \cap C^*$, we first pass to the ring $C(\beta X)$ by means of the canonical isomorphism $f \to f^\beta$ of $C^*(X)$ onto $C(\beta X)$ [3, 6.6(b)], and then we pass to the family of prime z-filters on βX [3, 2.12]. According to the Gelfand-Kolmogoroff theorem [3, 7.3], the prime ideal in $C(\beta X)$ corresponding to $M^p \cap C^*$ is given by

$$(M^p \cap C^*)^\beta = \{g \in C(\beta X) :\ p \in \mathrm{cl}_{\beta X} \boldsymbol{Z}_X(g \mid X)\}.$$

Since $\boldsymbol{Z}_X(g \mid X) = \boldsymbol{Z}_{\beta X}(g) \cap X$, this is a z-ideal in $C(\beta X)$. We denote the corresponding prime z-filter on βX by $\mathfrak{N}^p$; thus

$$\mathfrak{N}^p = \{Z \in \boldsymbol{Z}(\beta X) :\ p \in \mathrm{cl}_{\beta X}(Z \cap X)\}.$$

Also, the minimal prime ideal Q^β of $C(\beta X)$ corresponding to Q is a z-ideal [3, 14.7]; we denote the corresponding minimal prime z-filter on βX by $\mathfrak{Q}$. Let $Z \in \mathfrak{Q}$ and let V be any zero-set-neighborhood of p in βX. Since $\mathfrak{Q} \subseteq \mathfrak{M}^p_{\beta X}$ and $V \in \mathfrak{O}^p_{\beta X}$ we have $V \in \mathfrak{Q}$ [3, 7.15] and thus $V \cap Z \in \mathfrak{Q}$. Using the minimality of $\mathfrak{Q}$, we choose a zero-set W not in $\mathfrak{Q}$ such that $(V \cap Z) \cup W = \beta X$. If $V \cap Z$ has empty interior in βX, then W is dense in βX; so $W = \beta X$ and $W \in \mathfrak{Q}$, contradicting the choice of W. Hence $V \cap Z$ has nonempty interior in βX, and thus $(V \cap Z) \cap X \neq \emptyset$. This shows that every neighborhood of p in βX meets $Z \cap X$; hence $p \in \mathrm{cl}_{\beta X}(Z \cap X)$ and $Z \in \mathfrak{N}^p$. Thus $\mathfrak{Q} \subseteq \mathfrak{N}^p$ and it follows that $Q \subseteq M^p \cap C^*$.

Now assume that P contains no unit of C. Let $f \in P$ and let V be any zero-set-neighborhood of p in βX. Since $\boldsymbol{Z}[P^\beta]$ is a prime z-filter on βX contained in $\mathfrak{M}^p_{\beta X}$, we have $V \in \boldsymbol{Z}[P^\beta]$, so that $V \cap \boldsymbol{Z}(f^\beta) \in \boldsymbol{Z}[P^\beta]$ and thus also $V \cap \boldsymbol{Z}(f) \in \boldsymbol{Z}[P]$. Since P contains no unit of C, $V \cap \boldsymbol{Z}(f) \neq \emptyset$. Hence $p \in \mathrm{cl}_{\beta X} \boldsymbol{Z}(f)$, i.e., $f \in M^p$. Thus $P \subseteq M^p \cap C^*$. The converse is immediate, and the last statement then follows from the comparability.

REMARKS. The second part of the theorem generalizes [3, 7.9]: $M^{*p} = M^p \cap C^*$ if and only if M^{*p} contains no unit of C.

We also note that a nonunit of C in $\boldsymbol{M}^{*p}$ need not be contained in $\boldsymbol{M}^p\cap C^*$. For example, choose any function g in $C^*(\mathbf{R})$ that vanishes at infinity and has nonempty compact zero-set. Then g is a nonunit of C and for any $p\in\beta\mathbf{R}-\mathbf{R}$, we have $g\in\boldsymbol{M}^{*p}$ but $g\notin\boldsymbol{M}^p\cap C^*$. Whenever $f^\beta(p)=0$, $\boldsymbol{Z}(f)\neq\varnothing$, but $p\notin\mathrm{cl}_{\beta X}\boldsymbol{Z}(f)$, Theorem I shows that although $f\in\boldsymbol{M}^{*p}$ and f is a nonunit of C, $\boldsymbol{M}^{*p}$ contains no prime ideal that contains f and contains only nonunits of C.

3. **z-ideals in C^*.** As in C, a *z-ideal* in C^* is an ideal I that contains any function that belongs to the same maximal ideals as some function in I (see [**8**, p. 30] and [**3**, 2.7, 4A.5]). Thus the z-ideals of $C^*(X)$ are the ideals that correspond to z-ideals of $C(\beta X)$ under the isomorphism $f\rightarrow f^\beta$, and the family of all prime z-ideals of $C^*(X)$ is order-isomorphic with the family of all prime z-filters on βX [**3**, 2.12].

Every minimal prime ideal in C is a z-ideal [**3**, 14.7]. Thus a prime ideal in C is minimal if it contains no prime z-ideals. Also, if the prime z-ideals contained in a given maximal ideal of C form a chain, then *all* the prime ideals contained in that maximal ideal form a chain. By [**3**, 6.6(c)], prime ideals in C^* also have these properties.

4. **The isomorphism theorem.** In the case that $\beta X-X$ is a zero-set in βX (equivalently, that X is locally compact and σ-compact), the prime z-ideal structure of $C^*(X)$ may be described entirely in terms of prime z-ideals in the rings $C(X)$ and $C(\beta X-X)$. Some of the known results on the structure of these rings will be applied in §§6 and 7 to obtain information on the structure of $C^*(X)$.

When X is locally compact and σ-compact, there is a bounded unit of C that belongs to $\boldsymbol{M}^{*p}$ for every $p\in\beta X-X$; thus $\boldsymbol{M}^p\cap C^*\neq\boldsymbol{M}^{*p}$ if and only if $p\notin X$.

THEOREM II. *Let X be locally compact and σ-compact, and let $p\in\beta X$.*

(a) *The family of prime z-ideals of $C^*(X)$ contained in $\boldsymbol{M}^p\cap C^*$ is order-isomorphic with the family of prime z-ideals of $C(X)$ contained in $\boldsymbol{M}^p$.*

(b) *The family of prime z-ideals of $C^*(X)$ properly containing $\boldsymbol{M}^p\cap C^*$ (when $p\notin X$) is order-isomorphic with the family of prime z-ideals of $C(\beta X-X)$ contained in $\boldsymbol{M}^p_{\beta X-X}$.*

PROOF. We first place the prime z-ideals contained in $\boldsymbol{M}^{*p}$ in order-preserving correspondence with the prime z-filters on βX contained in $\mathfrak{M}^p_{\beta X}$, by means of the mapping $P\rightarrow\boldsymbol{Z}[P^\beta]$. Under this mapping, we have $\boldsymbol{M}^p\cap C^*\rightarrow\mathfrak{N}^p$ (see §2). The order-isomorphisms will now be obtained by means of traces and induced z-filters [**10**, §5].

(a) If $\mathcal{P}$ is a prime z-filter contained in $\mathfrak{N}^p$, then every member of

$\mathcal{P}$ meets X. By [**10**, Theorem 5.2], the trace

$$\mathcal{P}|X = \{Z \cap X: Z \in \mathcal{P}\}$$

of $\mathcal{P}$ on X is a prime z-filter on X. Since $\mathcal{P} \subseteq \mathfrak{N}^p$, we have $\mathcal{P}|X \subseteq \mathfrak{M}_X^p$. If $\mathcal{Q}$ is any prime z-filter on X contained in $\mathfrak{M}_X^p$, the induced prime z-filter

$$\mathcal{Q}^\# = \{Z \in \boldsymbol{Z}(\beta X): Z \cap X \in \mathcal{Q}\}$$

is clearly contained in $\mathfrak{N}^p$, and $\mathcal{Q}^\#|X = \mathcal{Q}$. Hence the mapping $\mathcal{P} \rightarrow \mathcal{P}|X$, for $\mathcal{P} \subseteq \mathfrak{N}^p$, is onto the family of prime z-filters on X contained in $\mathfrak{M}_X^p$. For any $\mathcal{P} \subseteq \mathfrak{N}^p$, it is immediate that $\mathcal{P} \subseteq (\mathcal{P}|X)^\#$. Now let $Z \in (\mathcal{P}|X)^\#$; thus there is $W \in \mathcal{P}$ such that $Z \cap X = W \cap X$. Since $W \subseteq Z \cup (\beta X - X)$, we have $Z \cup (\beta X - X) \in \mathcal{P}$. But $\beta X - X \notin \mathcal{P}$, so $Z \in \mathcal{P}$. Hence $\mathcal{P} = (\mathcal{P}|X)^\#$ and it follows that the mapping is one-to-one.

(b) If $\mathcal{P}$ is a prime z-filter on βX properly containing $\mathfrak{N}^p$, then by [**10**, Theorem 5.2], the trace $\mathcal{P}|(\beta X - X)$ is a prime z-filter on $\beta X - X$. Since $\mathcal{P} \subseteq \mathfrak{M}_{\beta X}^p$, we have $\mathcal{P}|(\beta X - X) \subseteq \mathfrak{M}_{\beta X - X}^p$. If $\mathcal{Q}$ is any prime z-filter on $\beta X - X$ contained in $\mathfrak{M}_{\beta X - X}^p$, the induced z-filter

$$\mathcal{Q}^\# = \{Z \in \boldsymbol{Z}(\beta X): Z \cap (\beta X - X) \in \mathcal{Q}\}$$

is prime and $\mathcal{Q}^\#|(\beta X - X) = \mathcal{Q}$. Since $\mathcal{Q} \subseteq \mathfrak{M}_{\beta X - X}^p$, we have $\mathcal{Q}^\# \subseteq \mathfrak{M}_{\beta X}^p$. Furthermore, the zero-set $\beta X - X$ is in $\mathcal{Q}^\#$ but clearly not in $\mathfrak{N}^p$; hence $\mathcal{Q}^\# \not\subseteq \mathfrak{N}^p$. It now follows from Theorem I that $\mathcal{Q}^\#$ properly contains $\mathfrak{N}^p$. Thus the mapping $\mathcal{P} \rightarrow \mathcal{P}|(\beta X - X)$, for $\mathfrak{N}^p \subset \mathcal{P}$, is onto the family of prime z-filters on $\beta X - X$ contained in $\mathfrak{M}_{\beta X - X}^p$. It is clear that $\mathcal{P} \subseteq (\mathcal{P}|(\beta X - X))^\#$. Now let $Z \in (\mathcal{P}|(\beta X - X))^\#$; thus there is $W \in \mathcal{P}$ such that $Z \cap (\beta X - X) = W \cap (\beta X - X)$. By Theorem I the z-ideal corresponding to $\mathcal{P}$ contains a unit of C, and thus $\beta X - X \in \mathcal{P}$. It follows that $Z \in \mathcal{P}$. Hence $\mathcal{P} = (\mathcal{P}|(\beta X - X))^\#$ and the mapping is one-to-one.

5. **An application to F-spaces.** An immediate consequence of Theorem II is a well-known theorem on F-spaces [2, 2.7], (see also [**3**, 14.27] and [**11**, 3.3]). A space T is an *F-space* if every finitely generated ideal in $C(T)$ is principal, or, equivalently, if the prime ideals contained in any given maximal ideal form a chain [**3**, 14.25]. In part (b) of Theorem II, the prime z-ideals properly containing the prime ideal $\boldsymbol{M}^p \cap C^*$ form a chain; thus we have

COROLLARY 1 (GILLMAN-HENRIKSEN). *If X is locally compact and σ-compact, then $\beta X - X$ is a compact F-space.*

6. **Immediate successors.** It was shown in [4, p. 432] that if X is locally compact and σ-compact, and $p\in\beta X-X$, then $\mathfrak{N}^p$ has an immediate successor $(\mathfrak{N}^p)^+$ in the family of prime z-filters on βX, generated by $\mathfrak{N}^p$ and the zero-set $\beta X-X$, i.e.

$$(\mathfrak{N}^p)^+ = (\mathfrak{N}^p, \beta X - X).$$

(This result may also be obtained from Theorem I, which shows that a prime z-filter contained in $\mathfrak{M}^p_{\beta X}$ properly contains $\mathfrak{N}^p$ if and only if it contains the zero-set $\beta X-X$.)

Furthermore, it is shown in [4, p. 433] that in the case of the countably infinite discrete space $\mathbf{N}$,

$$(\mathfrak{N}^p)^+ = (\mathcal{O}^p_{\beta\mathbf{N}}, \beta\mathbf{N} - \mathbf{N}).$$

We now generalize this as follows.

COROLLARY 2. *Let X be locally compact and σ-compact, and let $p\in\beta X-X$. Then*

$$(\mathfrak{N}^p)^+ = (\mathcal{O}^p_{\beta X}, \beta X - X).$$

Hence the immediate successor of $M^p\cap C^*$ in the family of prime z-ideals of $C^*(X)$ consists of all functions f such that f^β vanishes on a neighborhood of p in $\beta X-X$.

PROOF. According to the construction of the second isomorphism in §4, we have $(\mathfrak{N}^p)^+=(\mathcal{O}^p_{\beta X-X})^\#$, and it is easily verified that

$$(\mathcal{O}^p_{\beta X}, \beta X - X) = (\mathcal{O}^p_{\beta X-X})^\#.$$

REMARK. The present paper began with the observation that although $\mathcal{O}^p_{\beta\mathbf{R}}$ is usually *not* prime (see [10, Theorem 11.2]), the z-filter $(\mathcal{O}^p_{\beta\mathbf{R}}, \beta\mathbf{R}-\mathbf{R})$ is *always* prime, because of the above representation as an induced z-filter and the Gillman-Henriksen theorem of §5. This raised the question of its relation to $\mathfrak{N}^p$ and $(\mathfrak{N}^p, \beta\mathbf{R}-\mathbf{R})$.

7. **Remote points and P-points.** For any space X, a point p in βX is called a *remote point in βX* if every member of $\mathfrak{M}^p_X$ has nonempty interior (see [1]). When X is a metric space, *remote* generalizes *isolated*: a point p in X itself is a remote point in βX if and only if it is an isolated point of X. Also, if D is a discrete space, every point in βD is a remote point in βD. When X is a metric space with no isolated points, a point p in βX is a remote point in βX if and only if p is in the closure of no discrete subset of X (see [5, §23, VIII]).

Under the continuum hypothesis, the existence of remote points in $\beta\mathbf{R}$ was shown in [**1**]. It is shown in Theorem 11.2 of [**10**] (the proof given there for the real line is also valid for the case considered here) that *if X is a separable metric space and $p \in \beta X$, the following are equivalent*: (a) *The prime ideals contained in M_X^p form a chain.* (b) M_X^p *is a minimal prime ideal.* (c) *p is a remote point in βX.*

A point p of a space T is a *P-point* of T if every zero-set containing p is a neighborhood of p [**3**, 4L]; equivalently, if M_T^p is a minimal prime ideal [**3**, **14.12**]. Under the continuum hypothesis, there exist P-points of $\beta X - X$ whenever X is locally compact but not pseudocompact [**3**, 9M].

Assuming the continuum hypothesis, Donald Plank [**12**, Theorem 6.2] has recently discovered points in $\beta\mathbf{R}-\mathbf{R}$ that are both remote points in $\beta\mathbf{R}$ and P-points of $\beta\mathbf{R}-\mathbf{R}$, points that are remote points but not P-points, points that are P-points but not remote points, and also points that are neither. He has also shown that each of these four classes of points is a dense subset of $\beta\mathbf{R}-\mathbf{R}$ of cardinal 2^c. These points provide examples for the various types of prime ideal structure of C^* described below.

Applying Theorems I and II, and Theorem 11.2 of [**10**] (as stated above), we obtain the following relations between points in βX and the prime ideal structure of $C^*(X)$. (Corollary 4 generalizes [**4**, Theorem 3.10], which gives the result for the case $X = \mathbf{N}$.)

COROLLARY 3. *Let X be a locally compact, σ-compact metric space, and let $p \in \beta X$. Then the following conditions are equivalent.*

(a) *The prime ideals of C^* contained in M^{*p} form a chain.*

(b) *$M^p \cap C^*$ is a minimal prime ideal of C^*.*

(c) *p is a remote point in βX.*

COROLLARY 4. *Let X be locally compact and σ-compact, and let $p \in \beta X - X$. Then M^{*p} is the immediate successor of $M^p \cap C^*$ in the family of prime z-ideals of $C^*(X)$ if and only if p is a P-point of $\beta X - X$.*

COROLLARY 5. *Let X be a locally compact, σ-compact metric space, and let $p \in \beta X - X$. Then the family of prime z-ideals of C^* contained in M^{*p} consists of just the two ideals M^{*p} and $M^p \cap C^*$ if and only if p is both a remote point in βX and a P-point of $\beta X - X$.*

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