PRIME IDEAL STRUCTURE OF RINGS OF
BOUNDLED CONTINUOUS FUNCTIONS

MARK MANDELKER

Introduction. The order structure of the family of prime ideals in
the ring $C$ of all real-valued continuous functions on a topological
space has been extensively studied; in this paper we study the ana-
logous problem in the subring $C^*$ of bounded functions. The funda-
mental property of prime ideals in $C^*$ is the following.

Main Theorem. Let $M^*$ be any maximal ideal of $C^*$ and let $M$ be
the unique maximal ideal of $C$ such that the prime ideal $M \cap C^*$ is con-
tained in $M^*$. Then every prime ideal contained in $M^*$ is comparable
with $M \cap C^*$.

The proof involves topological properties of the Stone-Čech com-
pactification $\beta X$ of a completely regular Hausdorff space $X$.

Of special interest are the prime $z$-ideals of $C^*$. When $X$ is a locally
compact, $\sigma$-compact Hausdorff space, we show that the family of
prime $z$-ideals of $C^*(X)$ contained in $M^*$ is composed of two sub-
families, order-isomorphic with naturally corresponding families of
prime $z$-ideals in the rings $C(X)$ and $C(\beta X - X)$.

1. Preliminaries. We shall use the terminology and notation of the
Gillman-Jerison text [3]. Applying [3, Theorem 3.9], we immediately
reduce the problem of the prime ideal structure of $C^*(X)$, and its
relation to $C(X)$, to the case that $X$ is a completely regular Hausdorff
space. A basic property of prime ideals in rings of functions that will
be used several times is a theorem of Kohls ([9, Theorem 2.4], see
also [3, 14.8(a), 6.6(c)]: In the ring $C(X)$, and also in $C^*(X)$, the
prime ideals containing a given prime ideal form a chain.

The proof of the main theorem is based on Kohls’ result and the
following theorem ([10, 4.4]; cf. [7, 3.1] and [6, p. 112]): A prime
$z$-filter $Q$ on a space $T$ is minimal if and only if for every zero-set $Z$ in $Q$
there is a zero-set $W$ not in $Q$ such that $Z \cup W = T$.

We shall use $\mathfrak{M}_p$ and $\mathfrak{O}_p$ to denote the $z$-filters $Z[M_p]$ and $Z[O_p]$, respectively.

2. The main theorem. Under the reduction made in §1 to the case
of a completely regular Hausdorff space $X$, a maximal ideal of $C^*(X)$
corresponds to a point $p$ of $\beta X$ and is denoted $M^*_p$, and the maximal
ideal $M_p$ of $C(X)$ that corresponds to $p$ is the unique maximal ideal

Received by the editors July 28, 1967.

1432
of \( C(X) \) such that \( M^p \cap C^* \subseteq M^p \) (see [3, Chapter 7]). Thus the main theorem takes the following form.

**Theorem I.** Let \( p \in \beta X \). Every prime ideal \( P \) of \( C^*(X) \) contained in \( M^p \) is comparable with \( M^p \cap C^* \). Specifically, \( P \subseteq M^p \cap C^* \) if and only if \( P \) contains no unit of \( C \), while \( M^p \cap C^* \subseteq P \) if and only if \( P \) contains a unit of \( C \).

**Proof.** Let \( P \) be any prime ideal of \( C^*(X) \) with \( P \subseteq M^p \). Choose a minimal prime ideal \( Q \) with \( Q \subseteq P \). To prove that \( P \) and \( M^p \cap C^* \) are comparable, it suffices to show that \( Q \subseteq M^p \cap C^* \), for then \( P \) and \( M^p \cap C^* \) both contain the prime ideal \( Q \).

To show that \( Q \subseteq M^p \cap C^* \), we first pass to the ring \( C(\beta X) \) by means of the canonical isomorphism \( f \mapsto f^\beta \) of \( C^*(X) \) onto \( C(\beta X) \) [3, 6.6(b)], and then we pass to the family of prime \( z \)-filters on \( \beta X \) [3, 2.12]. According to the Gelfand-Kolmogoroff theorem [3, 7.3], the prime ideal in \( C(\beta X) \) corresponding to \( M^p \cap C^* \) is given by

\[
(M^p \cap C^*)^\beta = \{ g \in C(\beta X) : \ p \in \text{cl}_{\beta X} Z_X(g \mid X) \}.
\]

Since \( Z_X(g \mid X) = Z_{\beta X}(g) \cap X \), this is a \( z \)-ideal in \( C(\beta X) \). We denote the corresponding prime \( z \)-filter on \( \beta X \) by \( \mathfrak{F}^p \); thus

\[
\mathfrak{F}^p = \{ Z \in Z(\beta X) : \ p \in \text{cl}_{\beta X}(Z \cap X) \}.
\]

Also, the minimal prime ideal \( Q^\beta \) of \( C(\beta X) \) corresponding to \( Q \) is a \( z \)-ideal [3, 14.7]; we denote the corresponding minimal prime \( z \)-filter on \( \beta X \) by \( Q \). Let \( Z \in Q \) and let \( V \) be any zero-set-neighborhood of \( p \) in \( \beta X \). Since \( Q \subseteq \mathfrak{F}^p \) and \( V \subseteq \text{cl}_{\beta X} \), we have \( V \subseteq Q \) [3, 7.15] and thus \( V \cap Z \in \mathfrak{F}^p \). Using the minimality of \( Q \), we choose a zero-set \( W \) not in \( Q \) such that \( (V \cap Z) \cup W = \beta X \). If \( V \cap Z \) has empty interior in \( \beta X \), then \( W \) is dense in \( \beta X \); so \( W = \beta X \) and \( W \subseteq Q \), contradicting the choice of \( W \). Hence \( V \cap Z \) has nonempty interior in \( \beta X \), and thus \( (V \cap Z) \cap X \neq \emptyset \). This shows that every neighborhood of \( p \) in \( \beta X \) meets \( Z \cap X \); hence \( p \in \text{cl}_{\beta X}(Z \cap X) \) and \( Z \in \mathfrak{F}^p \). Thus \( Q \subseteq \mathfrak{F}^p \) and it follows that \( Q \subseteq M^p \cap C^* \).

Now assume that \( P \) contains no unit of \( C \). Let \( f \in P \) and let \( V \) be any zero-set-neighborhood of \( p \) in \( \beta X \). Since \( Z[P^\beta] \) is a prime \( z \)-filter on \( \beta X \) contained in \( \mathfrak{F}^p \), we have \( V \subseteq Z[P^\beta] \), so that \( V \cap Z(f^\beta) \subseteq Z[P^\beta] \) and thus also \( V \cap Z(f) \subseteq Z[P] \). Since \( P \) contains no unit of \( C \), \( V \cap Z(f) \neq \emptyset \). Hence \( p \in \text{cl}_{\beta X} Z(f) \), i.e., \( f \in M^p \). Thus \( P \subseteq M^p \cap C^* \). The converse is immediate, and the last statement then follows from the comparability.

**Remarks.** The second part of the theorem generalizes [3, 7.9]: \( M^{*p} = M^p \cap C^* \) if and only if \( M^{*p} \) contains no unit of \( C \).
We also note that a nonunit of $C$ in $M^p \cap C^*$ need not be contained in $M^p \cap C^*$. For example, choose any function $g$ in $C^*(\mathbb{R})$ that vanishes at infinity and has nonempty compact zero-set. Then $g$ is a nonunit of $C$ and for any $p \in \beta \mathbb{R} - \mathbb{R}$, we have $g \in M^p$ but $g \notin M^p \cap C^*$. Whenever $f^\beta(p) = 0$, $Z(f) \neq \emptyset$, but $p \notin \text{cl}_X Z(f)$, Theorem I shows that although $f \in M^p$ and $f$ is a nonunit of $C$, $M^p \cap C^*$ contains no prime ideal that contains $f$ and contains only nonunits of $C$.

3. z-ideals in $C^*$. As in $C$, a z-ideal in $C^*$ is an ideal $I$ that contains any function that belongs to the same maximal ideals as some function in $I$ (see [8, p. 30] and [3, 2.7, 4A.5]). Thus the z-ideals of $C^*(X)$ are the ideals that correspond to z-ideals of $C(\beta X)$ under the isomorphism $f \to f^\beta$, and the family of all prime z-ideals of $C^*(X)$ is order-isomorphic with the family of all prime z-filters on $\beta X$ [3, 2.12].

Every minimal prime ideal in $C$ is a z-ideal [3, 14.7]. Thus a prime ideal in $C$ is minimal if it contains no prime z-ideals. Also, if the prime z-ideals contained in a given maximal ideal of $C$ form a chain, then all the prime ideals contained in that maximal ideal form a chain. By [3, 6.6(c)], prime ideals in $C^*$ also have these properties.

4. The isomorphism theorem. In the case that $\beta X - X$ is a zero-set in $\beta X$ (equivalently, that $X$ is locally compact and $\sigma$-compact), the prime z-ideal structure of $C^*(X)$ may be described entirely in terms of prime z-ideals in the rings $C(X)$ and $C(\beta X - X)$. Some of the known results on the structure of these rings will be applied in §§6 and 7 to obtain information on the structure of $C^*(X)$.

When $X$ is locally compact and $\sigma$-compact, there is a bounded unit of $C$ that belongs to $M^p \cap C^*$ for every $p \in \beta X - X$; thus $M^p \cap C^* \neq M^p$ if and only if $p \notin X$.

**Theorem II.** Let $X$ be locally compact and $\sigma$-compact, and let $p \in \beta X$.

(a) The family of prime z-ideals of $C^*(X)$ contained in $M^p \cap C^*$ is order-isomorphic with the family of prime z-ideals of $C(X)$ contained in $M^p$.

(b) The family of prime z-ideals of $C^*(X)$ properly containing $M^p \cap C^*$ (when $p \notin X$) is order-isomorphic with the family of prime z-ideals of $C(\beta X - X)$ contained in $M^p_{\beta X - X}$.

**Proof.** We first place the prime z-ideals contained in $M^p \cap C^*$ in order-preserving correspondence with the prime z-filters on $\beta X$ contained in $\mathfrak{m}^p_{\beta X}$, by means of the mapping $P \to Z[P^\beta]$. Under this mapping, we have $M^p \cap C^* \to \mathfrak{m}^p$ (see §2). The order-isomorphisms will now be obtained by means of traces and induced z-filters [10, §5].

(a) If $\varnothing$ is a prime z-filter contained in $\mathfrak{m}^p$, then every member of
Φ meets X. By [10, Theorem 5.2], the trace
\[ Φ|X = \{ Z \cap X : Z ∈ Φ \} \]
of Φ on X is a prime z-filter on X. Since Φ ⊆ ℳ, we have Φ|X ⊆ ℳ. If Q is any prime z-filter on X contained in ℳ, the induced prime z-filter
\[ Q^* = \{ Z ∈ Z(βX) : Z \cap X ∈ Q \} \]
is clearly contained in ℳ, and Q^*|X = Q. Hence the mapping Φ → Φ|X, for Φ ⊆ ℳ, is onto the family of prime z-filters on X contained in ℳ. For any Φ ⊆ ℳ, it is immediate that Φ ⊆ (Φ|X)^. Now let Z ∈ (Φ|X)^; thus there is W ∈ Φ such that Z ∩ X = W ∩ X. Since W ⊆ Z ∪ (βX – X), we have Z ∪ (βX – X) ∈ Φ. But βX – X ∈ Φ, so Z ∈ Φ. Hence Φ = (Φ|X)^ and it follows that the mapping is one-to-one.

(b) If Φ is a prime z-filter on βX properly containing ℳ, then by [10, Theorem 5.2], the trace (Φ| (βX – X)) is a prime z-filter on βX – X. Since Φ ⊆ ℳ, we have Φ| (βX – X) ⊆ ℳ. If Q is any prime z-filter on βX – X contained in ℳ, the induced z-filter
\[ Q^* = \{ Z ∈ Z(βX) : Z \cap (βX – X) ∈ Q \} \]
is prime and Q^*| (βX – X) = Q. Since Q ⊆ ℳ, we have Q^* ⊆ ℳ. Furthermore, the zero-set (βX – X) is in Q^* but clearly not in ℳ; hence Q^* ⊆ ℳ. It now follows from Theorem I that Q^* properly contains ℳ. Thus the mapping Φ → Φ| (βX – X), for ℳ ⊆ Φ, is onto the family of prime z-filters on βX – X contained in ℳ. It is clear that Φ ⊆ (Φ| (βX – X))^*. Now let Z ∈ (Φ| (βX – X))^*; thus there is W ∈ Φ such that Z ∩ (βX – X) = W ∩ (βX – X). By Theorem I the z-ideal corresponding to Φ contains a unit of C, and thus βX – X ∈ Φ. It follows that Z ∈ Φ. Hence Φ = (Φ| (βX – X))^* and the mapping is one-to-one.

5. An application to F-spaces. An immediate consequence of Theorem II is a well-known theorem on F-spaces [2, 2.7], (see also [3, 14.27] and [11, 3.3]). A space T is an F-space if every finitely generated ideal in C(T) is principal, or, equivalently, if the prime ideals contained in any given maximal ideal form a chain [3, 14.25]. In part (b) of Theorem II, the prime z-ideals properly containing the prime ideal M^p ∩ C^* form a chain; thus we have

Corollary 1 (Gillman-Henriksen). If X is locally compact and σ-compact, then βX – X is a compact F-space.
6. Immediate successors. It was shown in [4, p. 432] that if $X$ is locally compact and $\sigma$-compact, and $p \in \beta X - X$, then $\mathcal{V}$ has an immediate successor $(\mathcal{V})^+$ in the family of prime $z$-filters on $\beta X$, generated by $\mathcal{V}$ and the zero-set $\beta X - X$, i.e.

$$(\mathcal{V})^+ = (\mathcal{V}, \beta X - X).$$

(This result may also be obtained from Theorem I, which shows that a prime $z$-filter contained in $\mathcal{M}_{\beta X}$ properly contains $\mathcal{V}$ if and only if it contains the zero-set $\beta X - X$.)

Furthermore, it is shown in [4, p. 433] that in the case of the countably infinite discrete space $\mathbb{N}$,

$$i_{01} = (\mathcal{O}_{\beta \mathbb{N}}, \beta \mathbb{N} - \mathbb{N}).$$

We now generalize this as follows.

**Corollary 2.** Let $X$ be locally compact and $\sigma$-compact, and let $p \in \beta X - X$. Then

$$(\mathcal{V})^+ = (\mathcal{O}_{\beta x}, \beta X - X).$$

Hence the immediate successor of $(\mathcal{V})^+ C^*$ in the family of prime $z$-ideals of $C^*(X)$ consists of all functions $f$ such that $f^p$ vanishes on a neighborhood of $p$ in $\beta X - X$.

**Proof.** According to the construction of the second isomorphism in \S4, we have $(\mathcal{V})^+ = (\mathcal{O}_{\beta x-x}^p)^+$, and it is easily verified that

$$(\mathcal{O}_{\beta x}, \beta X - X) = (\mathcal{O}_{\beta x-x}^p)^.$$

**Remark.** The present paper began with the observation that although $\mathcal{O}_{\beta \mathbb{R}}$ is usually not prime (see [10, Theorem 11.2]), the $z$-filter $(\mathcal{O}_{\beta \mathbb{R}}, \beta \mathbb{R} - \mathbb{R})$ is always prime, because of the above representation as an induced $z$-filter and the Gillman-Henriksen theorem of \S5. This raised the question of its relation to $\mathcal{V}$ and $(\mathcal{V}, \beta \mathbb{R} - \mathbb{R})$.

7. Remote points and $P$-points. For any space $X$, a point $p$ in $\beta X$ is called a remote point in $\beta X$ if every member of $\mathcal{M}_X$ has non-empty interior (see [1]). When $X$ is a metric space, remote generalizes isolated: a point $p$ in $X$ itself is a remote point in $\beta X$ if and only if it is an isolated point of $X$. Also, if $D$ is a discrete space, every point in $\beta D$ is a remote point in $\beta D$. When $X$ is a metric space with no isolated points, a point $p$ in $\beta X$ is a remote point in $\beta X$ if and only if $p$ is in the closure of no discrete subset of $X$ (see [5, \S23, VIII]).
Under the continuum hypothesis, the existence of remote points in \( \beta\mathbb{R} \) was shown in [1]. It is shown in Theorem 11.2 of [10] (the proof given there for the real line is also valid for the case considered here) that if \( X \) is a separable metric space and \( p \in \beta X \), the following are equivalent: (a) The prime ideals contained in \( M_p^\mathbb{X} \) form a chain. (b) \( M_p^\mathbb{X} \) is a minimal prime ideal. (c) \( p \) is a remote point in \( \beta X \).

A point \( p \) of a space \( T \) is a \( P \)-point of \( T \) if every zero-set containing \( p \) is a neighborhood of \( p \) [3, 4L]; equivalently, if \( M_p^\beta \) is a minimal prime ideal [3, 14.12]. Under the continuum hypothesis, there exist \( P \)-points of \( \beta X - X \) whenever \( X \) is locally compact but not pseudo-compact [3, 9M].

Assuming the continuum hypothesis, Donald Plank [12, Theorem 6.2] has recently discovered points in \( \beta\mathbb{R} - \mathbb{R} \) that are both remote points in \( \beta\mathbb{R} \) and \( P \)-points of \( \beta\mathbb{R} - \mathbb{R} \), points that are remote points but not \( P \)-points, points that are \( P \)-points but not remote points, and also points that are neither. He has also shown that each of these four classes of points is a dense subset of \( \beta\mathbb{R} - \mathbb{R} \) of cardinal 2\(^c\). These points provide examples for the various types of prime ideal structure of \( C^* \) described below.

Applying Theorems I and II, and Theorem 11.2 of [10] (as stated above), we obtain the following relations between points in \( \beta X \) and the prime ideal structure of \( C^*(X) \). (Corollary 4 generalizes [4, Theorem 3.10], which gives the result for the case \( X = \mathbb{N} \).)

**Corollary 3.** Let \( X \) be a locally compact, \( \sigma \)-compact metric space, and let \( p \in \beta X \). Then the following conditions are equivalent.

(a) The prime ideals of \( C^* \) contained in \( M_p^\mathbb{X} \) form a chain.

(b) \( M_p^\mathbb{X} \cap C^* \) is a minimal prime ideal of \( C^* \).

(c) \( p \) is a remote point in \( \beta X \).

**Corollary 4.** Let \( X \) be locally compact and \( \sigma \)-compact, and let \( p \in \beta X - X \). Then \( M_p^\mathbb{X} \) is the immediate successor of \( M_p^\mathbb{X} \cap C^* \) in the family of prime \( z \)-ideals of \( C^*(X) \) if and only if \( p \) is a \( P \)-point of \( \beta X - X \).

**Corollary 5.** Let \( X \) be a locally compact, \( \sigma \)-compact metric space, and let \( p \in \beta X - X \). Then the family of prime \( z \)-ideals of \( C^* \) contained in \( M_p^\mathbb{X} \) consists of just the two ideals \( M_p^\mathbb{X} \) and \( M_p^\mathbb{X} \cap C^* \) if and only if \( p \) is both a remote point in \( \beta X \) and a \( P \)-point of \( \beta X - X \).

**References**


*University of Kansas*