NONINJECTIVE CYCLIC MODULES

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In [3], it is shown that a ring $R$ such that every cyclic right $R$-module is injective must be semisimple Artin. In this note, that proof is greatly simplified, and it is shown that a hereditary ring cannot contain an infinite direct product of subrings.

$R$ will denote a ring with 1, all modules will be unital right $R$-modules, and all homomorphisms $R$-homomorphisms. For a module $M$, $E(M)$ will denote its injective hull (see [2]).

Theorem. Let $\{e_i | i \in \mathcal{S}\}$ be an infinite set of orthogonal idempotents of $R$. Assume for each $A \subseteq \mathcal{S}$, there exists $m_A \in R$ such that $m_A e_i = e_i$ for all $i \in A$, and $e_i m_A = 0$ for all $j \in \mathcal{S} - A$. Then for all $M_R \supseteq R_R$, $M/(\sum_{i \in \mathcal{S}} e_i R + \ker \pi)$ is not injective, where $\pi: R \twoheadrightarrow \prod_{i \in \mathcal{S}} e_i R$, $\pi(m) = (e_i m)$.

Proof. Let $\mathcal{S} = \bigcup_{A \in \mathcal{A}} A$, where $\mathcal{A}$ is infinite and for all $A$, $B \in \mathcal{A}$, $A$ is infinite and $A \cap B \neq \emptyset \Rightarrow A = B$. By Zorn's lemma, $\mathcal{A}$ can be enlarged to a set $\mathcal{B} \subseteq$ the power set of $\mathcal{S}$ maximal with respect to the properties

(i) for all $A \in \mathcal{B}$, $A$ is infinite, and
(ii) for all $A$ and $B$ in $\mathcal{B}$, $A \neq B \Rightarrow A \cap B$ is finite.

Let $\Sigma = \sum_{i \in \mathcal{S}} e_i R + \ker \pi.$ Then $\Sigma$ is precisely the set of elements of $R$ annihilated by almost all $e_i$. Let $A \in \mathcal{B}$, $r \in R$, and assume $m_A r \in \Sigma$. Then there exist an infinite number of $i$ (all in $A$) such that $e_i m_A r \neq 0$. For any set $\{A_j | 1 \leq j \leq n\} \subseteq \mathcal{B} - \{A\}$, $A \cap \bigcup_{j=1}^{n} A_j$ is finite. Thus for all but a finite number of $i \in A$, $e_i m_A = 0$ for all $j$, $1 \leq j \leq n$. Then $m_A r \in \sum_{j=1}^{n} m_{A_j} R + \Sigma$, so $\sum_{A \in \mathcal{B}} (m_A + \Sigma) R$ is direct in $M/\Sigma$.

Define $\phi: \sum_{A \in \mathcal{B}} (m_A R + \Sigma)/\Sigma \twoheadrightarrow M/\Sigma$ by

$$\phi(m_A) = m_A \quad A \in \mathcal{A},$$

$$= 0 \quad A \in \mathcal{B} - \mathcal{A}.$$

Assume $\phi$ extends to a homomorphism $\bar{\phi}$ from $R/\Sigma \twoheadrightarrow M/\Sigma$. Let $\bar{\phi}(1 + \Sigma) = m + \Sigma$. Then for all $A \in \mathcal{A}$, $m m_A = m_A + \sum_{i=1}^{n} e_i r_{l} + k$, so $A' = \{a \in A | e_a m e_a = e_a \} \supseteq A - \{i_l | 1 \leq l \leq n\}$ is infinite.

Let $C$ be a choice set for $\{A' | A \in \mathcal{A}\}$. By the maximality of $\mathcal{B}$, $C \cap D$ is infinite for some $D \in \mathcal{B}$, and $D$ cannot belong to $\mathcal{A}$. But then

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$mm_D \in \Sigma$, so for all but a finite number of $i \in \mathfrak{g}$, $e_i mm_D = 0$. Hence for all but a finite number of $d \in C \cap D$, $0 = e_d mm_D$, but for all $d \in C \cap D$, $e_d = e_d mm_D e_d$, a contradiction.

**Corollary.** Let $R$ contain an infinite ring direct product $\prod_{i \in \mathfrak{g}} R_i$, where $R_i$ is a ring with identity $e_i$. Then $R$ is not hereditary.

**Proof.** By [1, p. 14], a ring $R$ is hereditary if and only if every quotient of an injective module is injective. $\{e_i \mid i \in \mathfrak{g}\}$ are orthogonal idempotents, and the characteristic function of $A$ will serve as $m_A$ in the theorem. Then $E(R)/\Sigma$ is not injective.

**Corollary.** Let $R$ be a ring such that every cyclic $R$-module is injective. Then $R$ is semisimple Artin.

**Proof.** $R$ is von Neumann regular and self injective. For any set of orthogonal idempotents $\{e_i \mid i \in \mathfrak{g}\} \subseteq R$ and $A \subseteq \mathfrak{g}$, let $m_A$ be the projection of 1 on $E(\sum_{i \in A} e_i R) \subseteq R$. Clearly $m_A e_i = e_i$ for all $i \in A$. Let $j \in \mathfrak{g} - A$. Then $Re_j m_A = Re$ for some $e = e^2$. If $x = m_A e^2 \in \sum_{i \in A} e_i R$, then $e_i x = 0$ so $e_m A e = 0$, $e_i R = 0$, and finally $x = 0$. Since $m_A R$ is an essential extension of $\sum_{i \in A} e_i R$, $m_A e R = 0$. Then $e_j m_A = e m_A e = 0$. The theorem then shows that $\mathfrak{g}$ cannot be infinite, so $R$ is semisimple Artin (see [4]).

**References**


