# A NOTE ON FIXED-POINT-FREE SOLVABLE OPERATOR GROUPS

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Let $A$ be a finite group and let $n$ be the number of primes, including multiplicities, which divide the order of $A$. Suppose further that $G$ is a finite solvable group which admits $A$ as a fixed-point-free operator group and such that $(|A|, |G|)=1$. Under these assumptions it has been conjectured that $l(G)$, the nilpotent length of $G$, is at most $n$, while $l_p(G)$, the $p$-length of $G$, is at most $[(n+1)/2]$. These inequalities have been verified and shown to be best-possible in certain special cases ([**3**]–[**5**], [**1**]). It is the purpose of this note to show that these conjectured upper bounds are actually obtained provided $A$ is solvable. Specifically we will prove

THEOREM 1. *Let $A$ be a finite solvable group with $n$ the number of primes, including multiplicities, which divide $|A|$. Let $p$ be a prime not dividing $|A|$. Then there exists a finite solvable group $G$ of order prime to $|A|$ which admits $A$ as a fixed-point-free operator group and such that*

$$l(G) = n, \qquad l_p(G) = [(n+1)/2].$$

We now adopt the convention that all groups referred to are finite. $\pi'(A)$ denotes the set of primes which do not divide $|A|$ while $\psi(A)$ denotes the number of primes, including multiplicities, which divide $|A|$. Other than the following definition, the rest of the notation is standard.

DEFINITION. If $A$ is a group and $p$, $q$ are distinct primes in $\pi'(A)$, then $A$ is said to satisfy $E(p, q, n)$ if there exist two $p$, $q$-groups $G$ and $H$ such that

(i) $A$ operates in a fixed-point-free manner on both $G$ and $H$.

(ii) $l(G)=l(H)=n$.

(iii) $O_p(G)=O_q(H)=1$.

REMARK. (ii) and (iii) imply that

$$l_q(G) = l_p(H) = [(n+1)/2].$$

LEMMA. *If $A$ is cyclic of prime order and $p$, $q$ are distinct members of $\pi'(A)$, then $A$ satisfies $E(p, q, 1)$.*

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Received by the editors July 22, 1967.

[1] Research supported in part by a grant from the National Research Council of Canada.

PROOF. Let $K$ be a finite field of characteristic $q$ containing a primitive $|A|$th root of unity $\theta$. Let $G$ be the additive group of $K$ and let $A$ operate on $G$ by multiplication by $\theta$. $H$ is constructed in a similar manner.

THEOREM 2. *If $B<A$ and $B$ satisfies $E(p, q, n)$ for $p, q \in \pi'(A)$, then $A$ satisfies $E(p, q, n)$.*

PROOF. Let $G$ be a $p, q$-group on which $B$ operates in a fixed-point-free manner and such that $l(G)=n$ and $O_p(G)=1$. Let $x_1, x_2, \cdots, x_m$ be a complete set of representatives for the left cosets of $B$ in $A$. Next for $i=1, 2, \cdots, m$ let $G_i$ be the set of all ordered pairs $(g, i)$ where $g \in G$. Under the multiplication $(g_1, i)(g_2, i)=(g_1g_2, i)$, $G_i$ becomes a group isomorphic to $G$. Finally let $G^*=G_1 \times G_2 \times \cdots \times G_m$. If $x \in A$, define $(g, i)^x$ to be $(g^y, j)$ where $y \in B$ and $x_i x = y x_j$. In this way $G^*$ admits $A$ as a fixed-point-free operator group. Obviously $l(G^*)=n$ and $O_p(G^*)=1$.

THEOREM 3. *Let $B \trianglelefteq A$ and $B$ satisfy $E(p, q, m)$ while $A/B$ satisfies $E(p, q, n)$. Then $A$ satisfies $E(p, q, m+n)$.*

PROOF. Let $G$ be a $p, q$-group which admits $B$ as a fixed-point-free operator group and such that $l(G)=m$ and $O_p(G)=1$. Use the method in the proof of the last theorem to construct a group $G^*$ isomorphic to the direct product of $[A:B]$ copies of $G$ such that $A$ operates in a fixed-point-free manner on $G^*$. From the fact that $B$ is normal in $A$, we easily conclude that $B$ is fixed-point-free on $G^*$.

Now $G^*/F_{m-1}(G^*)$ is either a $p$- or a $q$-group. Thus there exists a $p, q$-group $H$ such that $A/B$ operates in a fixed-point-free manner on $H$, $l(H)=n$, and

$$(|G^*/F_{m-1}(G^*)|, |F_1(H)|) = 1.$$

We will consider $A$ as operating on $H$ by letting $B$ operate trivially.

Consider the group $G^* \sim H$, the wreath product of $G^*$ by $H$. $l(G^* \sim H)=m+n$ and $O_p(G^* \sim H)=1$ [2]. One way of representing $G^* \sim H$ is as follows: If $h \in H$, let $G_h^*=\{(g, h) | g \in G^*\}$, and define a multiplication in $G_h^*$ by $(g_1, h)(g_2, h)=(g_1g_2, h)$. Thus $G_h^*$ is a group isomorphic to $G^*$. Let $\overline{G}$ be the direct product of the $G_h^*$ for all $h \in H$. $H$ operates on $\overline{G}$ according to the rule $(g, h_1)^h=(g, h_1h)$. Then $G^* \sim H$ is the semidirect product $\overline{G}H$. $A$ operates on $\overline{G}$ as follows:

$$(g, h)^x = (g^x, h^x) \quad \text{if } x \in A.$$

It follows that $(g, h_1)^{x^{-1}hx}=(g, h_1)^{(h^x)}$ for $x \in A$, $h \in H$. Thus we may consider $A$ as operating on $\overline{G}H$.

Clearly $A$ is fixed-point-free on $\bar{G}H/\bar{G}\cong H$. If $y\in B$, then $(g, h)^y = (g^y, h)$. Since $B$ is fixed-point-free on $G^*$, we easily conclude that $B$ is fixed-point-free on $\bar{G}$. Thus $A$ is fixed-point-free on both $\bar{G}$ and $\bar{G}H/\bar{G}$. This finishes the proof.

COROLLARY. *If $A$ is solvable and $p$, $q$ are distinct members of $\pi'(A)$, then $A$ satisfies $E(p, q, \psi(A))$.*

PROOF. We use induction on $\psi(A)$. If $B$ is a maximal normal subgroup of $A$, then $|A/B|$ is a prime and $\psi(B) = \psi(A) - 1$. By induction $B$ satisfies $E(p, q, \psi(A)-1)$ while $A/B$ satisfies $E(p, q, 1)$. The theorem now implies that $A$ satisfies $E(p, q, \psi(A))$. Clearly Theorem 1 is an immediate consequence of this corollary.

## REFERENCES

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