AN EXTENSION OF A THEOREM OF E. STEIN

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Consider the singular integral

$$(1)\qquad (Tf)(x) = \text{P.V.} \int_{E_n} \frac{H(x, x-y)}{|x-y|^n} f(y)dy,$$

where E_n is the n-dimensional Euclidean space, and the integral is taken in the Lebesgue sense.

Calderón and Zygmund, [2], [3], have proved that under suitable conditions on the kernel $H(x, x-y)$

$$(2)\qquad \|Tf\|_p \leqq A\|f\|_p, \qquad 1 < p < \infty,$$

where A is a positive constant and $\|\cdot\|_p$ is the L_p norm. In the one-dimensional case Babenko [1] has shown that this inequality remains true if the measure dx is replaced by the measure $|x|^\beta dx$, and E. Stein [4] proved that in n dimensions

$$(3)\qquad \|(Tf)(x)\,|x|^\beta\|_p \leqq A_{p,\beta}\|f(x)\,|x|^\beta\|_p$$

for $1<p<\infty$ and $-n/p<\beta<n/p'$, $(1/p+1/p'=1)$.

He based his proof on the following

LEMMA (STEIN). *Let*

$$(4)\qquad K(x, y) = |1 - (|x|/|y|)^\beta| / |x-y|^n$$

and let

$$(5)\qquad (Uf)(x) = \int_{E_n} K(x, y)f(y)dy.$$

Then

$$\|(Uf)(x)\|_p \leqq A_{p,\beta}\|f(x)\|_p \qquad \text{if } -n/p < \beta < n/p'.$$

Stein's proof of (3) does not apply to $p=1$. In fact, the operator T is not of type (1, 1), i.e. it does not satisfy (2) for $p=1$. But T is of weak type (1, 1), that is, for some $s<1$, and every set X of finite measure

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(6) $$\left(\int_X |(Tf)(x)|^s dx\right)^{1/s} \leqq M |X|^{(1/s)-1} \|f\|_1,$$

where M is a positive constant, and $|X|$ is the Lebesgue measure of X. We will prove

THEOREM 1. *Let* $-n<\beta<0$, *and let* Uf *be defined by* (5). *Then* Uf *is of weak type* (1, 1).

And, as a consequence,

THEOREM 2. *Let*

$$(Tf)(x) = \text{P.V.} \int_{E_n} \frac{H(x, x-y)}{|x-y|^n} f(y)dy$$

and assume that T *is of weak type* (1.1), *and that* $|H(x, x-y)| \leqq B$. *Then*

$$\left(\int_X |(Tf)(x)|^s |x|^s dx\right)^{1/s} \leqq M |x|^{(1-s)/s} \|f(x)|x|^\beta\|_1$$

for every set X *of finite measure,* $-n<\beta<0$ *and* $(n-1)/n<s<1$.

PROOF OF THEOREM 1. We will prove that U satisfies (6) for any s such that

(7) $$(n-1)/n < s < 1.$$

Let $\lambda=|y|/|x|$. The kernel $K(x, y)$ is singular for $\lambda=0$, $\lambda=1$, $\lambda=\infty$. Define:

$K_1(x, y) = K(x, y)$ for $0 \leqq \lambda \leqq 1/2$, zero otherwise,

$K_2(x, y) = K(x, y)$ for $\lambda \geqq 2$, zero otherwise, and

$K_3(x, y) = K(x, y)$ for $1/2 < \lambda < 2$, zero otherwise,

and set

(8) $$(U_i f)(x) = \int_{E_n} K_i(x, y) f(y) dy, \qquad i = 1, 2, 3,$$

assuming, as we may, $f(x) \geqq 0$. Let ξ be the unit vector in the direction of x, $r=|x|$, η the unit vector in the direction of y and $R=|y|$. Let $d\omega_\xi$ and $d\omega_\eta$ be the elements of Euclidean measure on the spheres $|x|=1$ and $|y|=1$, and call Σ_ξ and Σ_η those unit spheres. Then (8) can be written

$$(U_i f)(x) = \int_{\Sigma_\eta}\int_0^\infty K_i(r\xi,\ R\eta)f(R\eta)R^{n-1}dR\,d\omega_\eta.$$

$K(x, y)$ is homogeneous of order $-n$, so the change of variable $R=\lambda r$ gives

$$(9)\qquad (U_i f)(x) = \int_{\Sigma_\eta}\int_0^\infty K_i(\xi,\ \lambda\eta)f(\lambda r\eta)\lambda^{n-1}d\lambda\, d\omega_\eta.$$

For $i=1$, we have

$$(10)\qquad K_1(\xi,\ \lambda\eta) \leqq \frac{|1-\lambda^{-\beta}|}{|1-2\lambda\cos(\xi,\ \eta)+\lambda^2|^{n/2}} \leqq A\,|1-\lambda^{-\beta}|$$

because $[1-2\lambda\cos\ (\xi,\ \eta)+\lambda^2]^{n-2}\geqq A>0$ if $0\leqq\lambda\leqq 1/2$, A being a constant. Therefore

$$(11)\qquad |(U_1 f)(r\xi)| \leqq A\int_{\Sigma_\eta}\int_0^{1/2}|1-\lambda^{-\beta}|\,\lambda^{n-1}f(\lambda r\eta)d\lambda\, d\omega_\eta.$$

We need an estimate for the integral

$$(12)\quad \left[\int_X |(U_1 f)(x)|^s dx\right]^{1/s} = \left[\int_{\Sigma_\xi}\int_{X_\xi}|(U_1 f)(r\xi)|^s r^{n-1}dr\,d\omega_\xi\right]^{1/s}$$

where, for each ξ in the unit sphere Σ_ξ, X_ξ is the set of r's such that $r\xi\in X$. By (11) and (12)

$$\left(\int_X |(U_1 f)(x)|^s dx\right)^{1/s}$$

$$\leqq A\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi}\left(\int_{\Sigma_\eta}\int_0^{1/2}|1-\lambda^{-\beta}|\,\lambda^{n-1}f(\lambda r\eta)d\lambda\, d\omega_\eta\right)^s \cdot r^{n-1}dr\right]d\omega_\xi\right\}^{1/s}$$

$$= A\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi}\left(\int_{\Sigma_\eta}\int_0^{1/2}|1-\lambda^{-\beta}|\,\lambda^{n-1}f(\lambda r\eta)r^{(n-1)/s}d\lambda\, d\omega_\eta\right)^s \cdot dr\right]d\omega_\xi\right\}^{1/s}$$

and by Hölder inequality, since $s<1$,

$$\left(\int_X |(U_1f)(x)|^s dx\right)^{1/s}$$

$$\leqq A\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi}\int_{\Sigma_\eta}\int_0^{1/2} |1-\lambda^{-\beta}|\,\lambda^{n-1}f(\lambda r\eta)r^{(n-1)/s}d\lambda d\omega_\eta dr\right]^s \cdot\left[\int_{X_\xi} dr\right]^{1-s} d\omega_\xi\right\}^{1/s}.$$

Since $0\leqq\lambda\leqq 1/2$, $\beta<0$, then $|1-\lambda^{-\beta}|\leqq 1$, and setting $l=\lambda r$ the above yields

$$\left(\int_X |(U_1f)(x)|^s dx\right)^{1/s}$$

$$\leqq A\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi}\int_{\Sigma_\eta}\int_0^{\infty} l^{n-1}f(l\eta)\,\frac{r^{(n-1)/s}}{r^{n-1}}\,\frac{dl}{r}\,d\omega_\eta dr\right]^s \cdot |X_\xi|^{1-s}d\omega_\xi\right\}^{1/s}, \tag{13}$$

where we have replaced the integral $\int_{X_\xi}dr$ by its value $|X_\xi|$. But the variables in the brackets are separated, so the right side of (13) is

$$A\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi} r^{(n-1-ns)/s}dr\right]^s\left[\int_{\Sigma_\eta}\int_0^\infty l^{n-1}f(l\eta)dld\omega_\eta\right]^s |X_\xi|^{1-s}d\omega_\xi\right\}^{1/s}.$$

But since $\int_{\Sigma_\eta}\int_0^\infty l^{n-1}f(l\eta)dld\omega_\eta=\|f\|_1$,

$$\left\{\int_X |(U_1f)(x)|^s dx\right\}^{1/s} \leqq A\|f\|_1\left\{\int_{\Sigma_\xi}|X_\xi|^{1-s}\left[\int_{X_\xi} r^{(n-1-ns)/s}dr\right]^s d\omega_\xi\right\}^{1/s}. \tag{14}$$

X being a set of finite measure, X_ξ can be infinite only in a set of measure zero. Thus, since by (7),

$$-1<(n-1-ns)/s<0$$

the integral in the inner bracket of (14) is convergent, and as the power of r is negative, it does not exceed

$$\int_0^{|X_\xi|} r^{(n-1-ns)/s}dr = s/(n-1-ns+s)\,|X_\xi|^{(n-1-ns+s)/s}.$$

Replacing in (14), and applying Hölder's inequality once more,

$$\left(\int_X |(U_1f)(x)|^s dx\right)^{1/s}$$

$$\leqq A_1\|f\|_1 \left\{\int_{\Sigma_\xi} |X_\xi|^{1-s}(|X_\xi|^{(n-1-ns+s)/s})^s d\omega_\xi\right\}^{1/s}$$

$$(15) \qquad = A_1\|f\|_1 \left\{\int_{\Sigma_\xi} |X_\xi|^{n(1-s)} d\omega_\xi\right\}^{1/s}$$

$$\leqq A_1\|f\|_1 \left(\int_{\Sigma_\xi} |X_\xi|^n d\omega_\xi\right)^{(1-s)/s}\left(\int_{\Sigma_\xi} d\omega_\xi\right)$$

$$\leqq M_1\|f\|_1 |X|^{(1-s)/s}.$$

Consider now $U_2 f$. Here $2 \leqq \lambda < \infty$, so

$$(16) \quad (U_2 f)(r\xi) = \int_{\Sigma_\eta}\int_2^\infty \frac{|1-\lambda^{-\beta}|\lambda^{n-1}}{|1-2\lambda\cos(\xi,\eta)+\lambda^2|^{n/2}} f(\lambda r\eta)d\lambda d\omega.$$

In this case we get

$$(17) \qquad \frac{|1-\lambda^{-\beta}|}{|1-2\lambda\cos(\xi,\eta)+\lambda^2|^{n/2}} \leqq B\frac{|1-\lambda^{-\beta}|}{\lambda^n}.$$

We set

$$\epsilon = (n-1)(1-s) \qquad (n-1-\epsilon)/s = n-1$$

$$(18) \quad I = \left\{\int_X |(U_2f)(x)|^s dx\right\}^{1/s} = \left\{\int_{\Sigma_\xi}\int_{X_\xi} |(U_2f)(r\xi)|^s r^{n-1} dr d\omega_\xi\right\}^{1/s}.$$

Using (17) in (16) and replacing above,

$$I \leqq B\left\{\int_{\Sigma_\xi}\left[\int_{X_\xi}\left(\int_{\Sigma_\eta}\int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda} f(\lambda r\eta)d\lambda d\omega_\eta\right)^s r^{n-1}dr\right]d\omega_\xi\right\}^{1/s}$$

$$= B\left\{\int_{\Sigma_\xi}\int_{X_\xi}\left(\int_{\Sigma_\eta}\int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda} f(\lambda r\eta) r^{(n-1-\epsilon)/s} d\lambda d\omega_\eta\right)^s \cdot r^\epsilon dr\right] d\omega_\xi\right\}^{1/s},$$

and by Hölder's inequality

$$I \leqq B \left\{ \int_{\Sigma_\xi} \left[\int_{X_\xi} \int_{\Sigma_\eta} \int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda} f(\lambda r\eta) r^{(n-1-\epsilon)/s} d\lambda d\omega_\eta dr \right]^s \cdot \left[\int_{X_\xi} r^{\epsilon/(1-s)} dr \right]^{1-s} d\omega_\xi \right\}^{1/s},$$

which by the change of integration variable $\rho = r\lambda$, becomes

$$I \leqq B \left\{ \int_{\Sigma_\xi} \left[\int_0^\infty \int_{\Sigma_\eta} \int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda} f(\rho\eta) \frac{\rho^{(n-1-\epsilon)/s}}{\lambda^{(n-1-\epsilon)/s}} d\lambda d\omega_\eta \frac{d\rho}{\lambda} \right]^s \cdot \left[\int_{X_\xi} r^{\epsilon/(1-s)} dr \right]^{1-s} d\omega_\xi \right\}^{1/s}.$$

In the first bracket the variables are separated so the right member above can be given the form

$$B \left\{ \int_{\Sigma_\xi} \left(\int_{X_\xi} r^{\epsilon/(1-s)} dr \right)^{1-s} \left(\int_0^\infty \int_{\Sigma_\eta} f(\rho\eta) \rho^{(n-1-\epsilon)/s} d\omega_\eta d\rho \right)^s \cdot \left(\int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda^{2+(n-1-\epsilon)/s}} d\lambda \right)^s d\omega_\xi \right\}^{1/s}. \tag{19}$$

Recalling (18) and noticing $\int_0^\infty \int_{\Sigma_\eta} f(\rho\eta)\rho^{n-1} d\omega_\eta d\rho = \|f\|_1$, we obtain

$$I \leqq B\|f\|_1 \left\{ \int_{\Sigma_\xi} \left(\int_{X_\xi} r^{n-1} dr \right)^{1-s} \left(\int_2^\infty \frac{|1-\lambda^{-\beta}|}{\lambda^{n+1}} d\lambda \right)^s d\omega_\xi \right\}^{1/s}. \tag{20}$$

As $\beta > -n$, the integral with regard to λ is convergent, and

$$\left(\int_X |(U_2 f)(x)|^s dx \right)^{1/s} \leqq \text{Const } \|f\|_1 \left[\int_{\Sigma_\xi} \left(\int_{X_\xi} r^{n-1} dr \right)^{1-s} d\omega_\xi \right]^{1/s}$$
$$\leqq \text{Const } \|f\|_1 \left\{ \left(\int_{\Sigma_\xi} \int_{X_\xi} r^{n-1} dr d\omega_\xi \right)^{1-s} \left(\int_{\Sigma_\xi} d\omega_\xi \right)^s \right\}^{1/s}$$

or

$$\left[\int_X |(U_2 f)(x)|^s dx \right]^{1/s} \leqq M_2 \|f\|_1 |X|^{(1-s)/s}. \tag{21}$$

In the case of U_3, Stein's proof applies, and U_3 is L_1 bounded, and a fortiori of weak type (1, 1). Thus with (15) and (21) the inequality (6) is established.

PROOF OF THEOREM 2. By hypothesis

$$\left\{\int_X \left| T[f(y)\,|y|^\beta]\right|^s dx\right\}^{1/s} \leqq M_1 \left| x \right|^{(1-s)/s} \| f(y)\,|y|^\beta \|_1.$$

It suffices to prove that

$$\int_X | T[f(y)\,|y|^\beta] - |x|^\beta T[f(y)]\,|^s dx\}^{1/s} \leqq M_1 |\, X\,|^{(1-s)/s} \| f(y)\,|\,y\,|^\beta \|_1.$$

We have

$$
\begin{aligned}
&| T[f(y)\,|y|^\beta] - |x|^\beta T[f(y)]\,| \\
&= \left| \int_{E_n} \frac{H(x, x-y)}{|x-y|^n} f(y)\,|y|^\beta dy - \int_{E_n} \frac{H(x, x-y)}{|x-y|^n} f(y)\,|x|^\beta dy \right| \\
&= \left| \int_{E_n} H(x, x-y)\, \frac{|y|^\beta - |x|^\beta}{|x-y|^n} f(y) dy \right| \\
&\leqq C \int_{E_n} \frac{|1 - |x|^\beta / |y|^\beta|}{|x-y|^n}\, |y|^\beta f(y) dy \\
&= C \int_{E_n} K(x, y)\,|y|^\beta f(y) dy.
\end{aligned}
$$

An application of Theorem 1 now yields the desired result.

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