# CONSTRUCTION OF RIEMANNIAN COVERINGS[1]

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1. **Introduction.** Ambrose's theorem on parallel translation of curvature [1] (stated in §3) says essentially that two suitably related Riemannian manifolds have a common Riemannian covering manifold. Our aim is to derive this theorem from general results on coverings that may be useful elsewhere. Proposition 1, below, formalizes the well-known heuristic scheme in which elements of a covering (by open sets) of a manifold $M$ are glued together to give a new manifold $X$ mapping onto $M$. Proposition 2 records the obvious way to assemble local mappings on $M$ into a global mapping of $X$. We carry Riemannian structures along, so that these mappings are local isometries. Proposition 3 gives a criterion for $X$, if connected, to be complete; thus the local isometries become Riemannian coverings. Our approach differs from Ambrose's mostly in that its atoms are (convex) open sets rather than broken geodesics.

2. **Riemannian coverings.** Let $\mathfrak{U}=\{U_\alpha | \alpha\in A\}$ be an indexed collection of subsets of some set. By a *semiequivalence relation* on the index set $A$ we mean a reflexive, symmetric relation $\sim$ such that (1) $\alpha\sim\beta$ and $\beta\sim\gamma$ imply $\alpha\sim\gamma$ whenever $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$, and (2) $\alpha\sim\beta$ in $A$ implies $U_\alpha\cap U_\beta\neq\emptyset$. (This last condition turns out to be merely a notational convenience.)

PROPOSITION 1. *Let $\mathfrak{U}=\{U_\alpha | \alpha\in A\}$ be an open covering of a Riemannian manifold $M$, and let $\sim$ be a semiequivalence relation on $A$. Then there exist* (1) *a Riemannian manifold $X$*, (2) *a local isometry $\psi: X\to M$, and* (3) *for each $\alpha\in A$, a cross-section $\lambda_\alpha: U_\alpha\to X$ of $\psi$ on $U_\alpha$ such that $\lambda_\alpha=\lambda_\beta$ on $U_\alpha\cap U_\beta\neq\emptyset$ if and only if $\alpha\sim\beta$ in $A$.*

PROOF. Let $D$ be the disjoint union of the elements of $\mathfrak{U}$, with obvious topology. If $p\in U_\alpha\subset M$, write $p_\alpha$ for the corresponding element of $D$. Define a relation $\approx$ on $D$ by: $p_\alpha\approx q_\beta$ provided $p=q$ and $\alpha\sim\beta$. Clearly, $\approx$ is an equivalence relation. Let $X$ be the quotient space $D/\approx$, with natural mapping $\pi: D\to X$. Let $U^\alpha$ denote $U_\alpha$ considered as an (open) subset of $D$; that is, $U^\alpha=\{p_\alpha | p\in U_\alpha\subset M\}$. We now prove

(1) $\pi$ is an open mapping. It suffices to show that if $O^\alpha$ is open in $U^\alpha$, then $\pi(O^\alpha)$ is open in $X$. For each index $\beta$ in $A$, let $P_\beta$ be the

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homeomorphism of $U^\beta$ onto $U_\beta$ sending each $p_\beta$ to $p$. Then the set $O^{\alpha\beta}=P_\beta^{-1}(P_\alpha(O^\alpha)\cap U_\beta)$ is open in $D$. But it is easy to check that $\pi^{-1}(\pi(O^\alpha))=\bigcup\{O^{\alpha\beta}\mid\beta\sim\alpha\}$. Since this set is open in $D$, $\pi(O^\alpha)$ is open in $X$.

(2) For each $\alpha\in A$, $\pi\mid U^\alpha$ is a homeomorphism onto an open set in $X$. It suffices to show that $\pi$ is one-one on $U^\alpha$. But $\pi(p_\alpha)=\pi(q_\alpha)$ means $p_\alpha\approx q_\alpha$; hence $p=q$ and $p_\alpha=q_\alpha$.

Because $\pi(p_\alpha)=\pi(q_\beta)$ implies $p=q$, there is a well-defined function $\psi: X\to M$ such that $\psi(\pi p_\alpha)=p$ for all $p_\alpha\in D$. It follows from (2) that $\psi$ restricted to $\pi(U^\alpha)$ is a homeomorphism onto $U_\alpha$. Let $\lambda_\alpha$: $U_\alpha\to\pi(U^\alpha)$ be the inverse homeomorphism.

(3) $\lambda_\alpha=\lambda_\beta$ on $U_\alpha\cap U_\beta\neq\emptyset$ if and only if $\alpha\sim\beta$ in $A$. If $\alpha\sim\beta$, then for $p\in U_\alpha\cap U_\beta$ we have $\lambda_\alpha(p)=\pi(p_\alpha)=\pi(p_\beta)=\lambda_\beta(p)$. Conversely, since we assume there is a point $p$ in $U_\alpha\cap U_\beta$, we have $\pi(p_\alpha)=\lambda_\alpha(p)=\lambda_\beta(p)=\pi(p_\beta)$; hence $\alpha\sim\beta$.

(4) $X$ is a Hausdorff space. Suppose $x\neq y$ in $X$. If $\psi(x)\neq\psi(y)$, then since $M$ is Hausdorff and $\psi$ continuous, $x$ and $y$ have disjoint neighborhoods in $X$. If $\psi(x)=\psi(y)=p\in M$, there exist indices $\alpha$ and $\beta$ such that $\pi(p_\alpha)=x$ and $\pi(p_\beta)=y$. Since $x\neq y$, we have $p_\alpha\not\approx p_\beta$, hence $\alpha\not\sim\beta$. Thus $\pi(U_\alpha)$ and $\pi(U_\beta)$ are disjoint neighborhoods of $x$ and $y$.

Since the open sets $\pi(U^\alpha)$, $\alpha\in A$, cover $X$, $\psi$ is a local homeomorphism of a Hausdorff space onto a Riemannian manifold. It follows automatically that there exist unique differentiable and Riemannian structures on $X$ such that $\psi$ is a local isometry.

PROPOSITION 2. *With hypotheses and notation as in Proposition* 1, *let* $\{\phi_\alpha: U_\alpha\to N\mid\alpha\in A\}$ *be a collection of local isometries into a Riemannian manifold N. If* $\alpha\sim\beta$ *in A implies* $\phi_\alpha=\phi_\beta$ *on* $U_\alpha\cap U_\beta$, *then there exists a unique local isometry* $\phi: X\to N$ *such that* $\phi\circ\lambda_\alpha=\phi_\alpha$ *for all* $\alpha\in A$.

PROOF. The collection of local isometries constitutes a continuous mapping $\Phi: D\to N$ of the disjoint union $D$. If $p_\alpha\approx q_\beta$, then $p=q$ and $\alpha\sim\beta$, hence $\Phi(p_\alpha)=\phi_\alpha(p)=\phi_\beta(q)=\Phi(q_\beta)$. Thus $\Phi$ is constant on equivalence classes modulo $\approx$. Then there results a continuous mapping $\phi: X\to N$ such that $\phi(\pi p_\alpha)=\Phi(p_\alpha)=\phi_\alpha(p)$ for all $p_\alpha\in D$. Since $\pi(p_\alpha)=\lambda_\alpha(p)$, it follows that $\phi=\phi_\alpha\circ\psi$ on each open set $\lambda_\alpha(U_\alpha)=\pi(U^\alpha)$. Thus $\phi$ is a local isometry.

Given $(\mathfrak{U}, \sim)$ and $X$ as in Proposition 1, let $\psi': X'\to M$ be another local isometry onto $M$ with cross-sections as in (3) of Proposition 1. Using Proposition 2 it is easy to see that there is a structure-preserving isometry from $X$ to $X'$.

Even if $M$ is connected, $X=X(\mathfrak{U}, \sim)$ need not be. However, define a *chain* in the index set $A$ of $\mathfrak{U}$ to be a finite sequence $\alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_n$ of successively related indices. Then if the elements of $\mathfrak{U}$ are connected and any two elements of $A$ are chainable, both $X$ and $M$ are connected.

We now find a criterion for the completeness of $X = X(\mathfrak{U}, \sim)$—at least when $X$ is connected. This derives from the Hopf-Rinow criterion that geodesics emanating from a single point be infinitely extendable. A chain $\alpha_1 \sim \cdots \sim \alpha_n$ in $A$ *covers* a curve segment $\sigma$: $[0, b] \to M$ provided there exist numbers $0 = t_0 < t_1 < \cdots < t_n = b$ such that $\sigma | [t_{i-1}, t_i]$ lies in $U_{\alpha_i}$ for $1 \leqq i \leqq n$. Then we say that $(\mathfrak{U}, \sim)$ is *geodesically extendable from a point* $p \in U_\alpha$ provided that any geodesic segment $\gamma$: $[0, b] \to M$ such that $\gamma(0) = p$ can be covered by a chain $\alpha = \alpha_1 \sim \cdots \sim \alpha_n$ in the index set of $\mathfrak{U}$.

PROPOSITION 3. *Let* $\mathfrak{U} = \{ U_\alpha | \alpha \in A \}$ *be an open covering of a connected Riemannian manifold* $M$, *with* $\sim$ *a semiequivalence relation on* $A$. *If* $M$ *is complete and* $(\mathfrak{U}, \sim)$ *is extendable from* $p \in U_\alpha$, *then in* $X = X(\mathfrak{U}, \sim)$ *the component* $C$ *containing the point* $\lambda_\alpha(p)$ *is complete. Hence* $\psi | C$: $C \to M$ *is a Riemannian covering.*

PROOF. Let $\beta$: $[0, b) \to C$ be a geodesic such that $\beta(0) = \lambda_\alpha(p)$. By the Hopf-Rinow theorem it suffices to show that $\beta$ has an extension past $b$. Since $\psi$ is a local isometry, $\psi \circ \beta$ is a geodesic. But $M$ is complete, so $\psi \circ \beta$ has a geodesic extension $\gamma$: $[0, b] \to M$. By hypothesis, $\gamma$ is covered by a chain $\alpha = \alpha_1 \sim \cdots \sim \alpha_n$; that is, there are numbers $t_i$ such that $\gamma | [t_{i-1}, t_i]$ lies in the domain $U_{\alpha_i}$ of $\lambda_{\alpha_i}$ $(1 \leqq i \leqq n)$. Thus we have well-defined geodesics $\lambda_{\alpha_i} \circ \gamma$: $[t_{i-1}, t_i] \to X$. And since $\alpha_{i-1} \sim \alpha_i$ we have $\lambda_{\alpha_{i-1}} = \lambda_{\alpha_i}$ on $U_{\alpha_{i-1}} \cap U_{\alpha_i}$; hence these segments constitute a single geodesic segment $\tilde{\beta}$: $[0, b] \to X$. By construction, $\beta$ and $\tilde{\beta}$ are initially the same, hence $\tilde{\beta}$ provides the required geodesic extension of $\beta$ to (thus past) $b$. The final assertion in the Proposition follows from a well-known theorem.

3. **Ambrose's theorem.** We formulate this result in the notation of [2]. In this section all manifolds are assumed to be connected.

Fix a point $p$ of a Riemannian manifold $M$. If $v$ is an element of the tangent space $M_p$, let $\gamma_v$: $[0, 1] \to M$ be the geodesic with initial velocity $v$, and let $\tau_v$: $M_p \to M_{\gamma_v(1)}$ be parallel translation along $\gamma_v$. If $v, w \in M_p$, let $\gamma_{vw}$: $[0, 2] \to M$ be the broken geodesic such that $\gamma_{vw}(t) = \gamma_v(t)$ for $t \in [0, 1]$, and $\gamma_{vw}(t) = \gamma_{\tau_v w}(t-1)$ for $t \in [1, 2]$. Let $\tau_{vw}$: $M_p \to M_{\gamma_{vw}(2)}$ be parallel translation along $\gamma_{vw}$.

THEOREM (AMBROSE [1]). *Let* $M$ *and* $N$ *be complete Riemannian manifolds; let* $p \in M$ *and* $q \in N$; *let* $\ell$: $M_p \to N_q$ *be a linear isometry.*

*If for each $v$, $w \in M_p$ the linear isometry $\ell_{vw} = \tau \ell_{v,\ w} \circ \ell \circ \tau_{vw}^{-1}$ preserves sectional curvature, then there exist* (1) *a complete Riemannian manifold $X$,* (2) *Riemannian coverings $\psi: X \to M$ and $\phi: X \to N$, and* (3) *a point $x \in X$ such that $\psi(x) = p$, $\phi(x) = q$, and $\ell \circ \psi_* = \phi_*$ at $x$.*

Thus if $M$ [also $N$] is simply connected, then $\phi \circ \psi^{-1}: M \to N$ is a Riemannian covering [isometry] with differential map $\ell$ at $p$.

PROOF. Fix a real-valued function $f > 0$ on $M$ such that for each $m \in M$ the open ball $B$ of radius $f(m)$ at $m$ is convex and normal. Normality means that $B$ is the diffeomorphic image, under $\exp_m$, of the ball of the same radius at 0 in $M_m$. Our only use of convexity is to conclude that the intersection of two such balls $B$ is connected.

For each $v \in M_p$, let $U(v)$ be the open ball at $\gamma_v(1)$ of radius $f(\gamma_v(1))$. Consider the linear isometry $\ell_v = \tau \ell_v \circ \ell \circ \tau_v^{-1}$ from $M_{\gamma_v(1)}$ to $N_{\gamma \ell_v(1)}$. Since $\ell_{vw}$ preserves sectional curvature for all $w$, it follows that $\ell_v$ and $U(v)$ satisfy the hypothesis of the Cartan Lemma, stated below. Its conclusion then gives a unique local isometry $\phi_v: U(v) \to N$ such that $\phi_{v*}$ at $\gamma_v(1)$ is $\ell_v$. Thus $\mathfrak{U} = \{U(v) \mid v \in M_p\}$ is a convex open covering of $M$, and we define a relation $\sim$ on its index set by: $v \sim w$ provided $\phi_v = \phi_w$ on $U(v) \cap U(w) \neq \emptyset$. Then $\sim$ is a semiequivalence relation, for if $u \sim v$, $v \sim w$, and $P = U(u) \cap U(v) \cap U(w) \neq \emptyset$, then $\phi_u = \phi_w$ on $P$, hence on the connected set $U(u) \cap U(w)$.

We assert that $(\mathfrak{U}, \sim)$ is geodesically extendable from $p \in U_0$ $(0 \in M_p)$. Let $\gamma: [0, b] \to M$ be a geodesic segment such that $\gamma(0) = p$. If $v$ is the initial velocity of $\gamma$, note that for $s \in [0, b]$ we have $\gamma(s) = \gamma_{sv}(1) \in U(sv)$. Thus $s$ has a connected neighborhood $I_s$ in $[0, b]$ such that $\gamma(I_s) \subset U(sv)$. It follows that there exist numbers $0 = t_0 = s_1 < t_1 < \cdots < s_n = t_n = b$ such that $\gamma | [t_{i-1}, t_i]$ lies in $U(s_i v)$ for $1 \leqq i \leqq n$. It remains to show that $s_i v \sim s_{i+1} v$. Write $\phi_i$ for $\phi_{s_i v}: U(s_i v) \to N$. Thus we must prove that $\phi_i = \phi_{i+1}$ on the connected open set $U(s_i v) \cap U(s_{i+1} v)$. By construction the differential map of $\phi_i$ at $\gamma_{s_i v}(1) = \gamma(s_i)$ is $\ell_{s_i v}$. Note that $\gamma | [s_i, t_i]$ lies in the domain $U(s_i v)$ of $\phi_i$. Since the differential maps of $\phi_i$ commute with parallel translation, it follows that $\phi_{i*}$ at $\gamma(t_i) = \gamma_{t_i v}(1)$ is $\ell_{t_i v}$. Similarly, $\gamma | [t_i, s_{i+1}]$ lies in $U(s_{i+1} v)$, and $\phi_{i+1*}$ at $\gamma(t_i)$ is also $\ell_{t_i v}$. Thus the two local isometries $\phi_i$ and $\phi_{i+1}$ agree on the connected set $U(s_i v) \cap U(s_{i+1} v)$.

The preceding argument also shows that any index $v$ is chainable to $0 \in M_p$. Thus, by an earlier remark, $X = X(\mathfrak{U}, \sim)$ is connected. By Proposition 3, $X$ is complete and $\psi: X \to M$ is a Riemannian covering. By Proposition 2, the local isometries $\phi_v$, $v \in M_p$, give rise to a local isometry $\phi: X \to N$, and $\phi$ is also a Riemannian covering. Let $x = \lambda_0(p)$, where $\lambda_0$ is the cross-section of $\psi$ on $U(0)$. We know that

$\phi \circ \lambda_0 = \phi_0$. But $\phi_{0*}$ at $p$ is $\ell$, and $\lambda_{0*}$ at $p$ is the inverse of $\psi_*$ at $x$. Hence $\phi_* = \ell \circ \psi_*$ at $x$.

LEMMA (CARTAN). *Let $U$ be a normal ball at a point $m$ in a Riemannian manifold $M$, let $N$ be a complete Riemannian manifold, and let $\ell: M_m \to N_q$ be a linear isometry. If $\ell_w = \tau_{\ell w} \circ \ell \circ \tau_w^{-1}$ preserves sectional curvature for all $w \in M_m$ of length less than the radius of $U$, then there exists a unique local isometry $\phi: U \to N$ such that $\phi_*$ at $m$ is $\ell$.*

This result is usually stated so as to give an isometry onto a corresponding normal ball in $N$. A proof can easily be deduced (without explicit use of the structural equations) from the following general fact: let $m \in M$, $x$ tangent to $M_m$ at $v$. Then $\exp_{m*}(x) = X(1)$, where $X$ is the unique Jacobi field on $\gamma_v$ such that $X(0) = 0$ and $X'(0) \in M_m$ corresponds canonically to $x \in (M_m)_v$.

## BIBLIOGRAPHY

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