SUBSETS OF FIRST COUNTABLE SPACES

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I. Summary. Let $X$ be a topological space. If $X$ is second countable, it is known [3, Theorem VIII, p. 246] that any sequence of subsets of $X$ contains a subsequence which is topologically convergent. In §II of this note we show that if $X$ is first countable, any net of subsets of $X$ contains a subnet which is topologically convergent. (See §II for definitions.)

Let $Q$ denote the space of compact nonempty subsets of a Hausdorff space $X$, topologized with the finite topology. In §III we characterize the spaces $X$ for which $Q$ is first countable. It previously had been thought that the first countability of $X$ was sufficient to imply first countability of $Q$ [4, Proposition 4.5.3, p. 162]. Appropriate counter examples are found in §III.

II. Topological convergence of sets. Let $\{A_j: j \in J\}$ be a net of subsets (not necessarily closed or nonempty) of a topological space $X$. 

$$\lim S(A_j) = \{x \in X: \text{for every neighborhood } U \text{ of } x \text{ and any } k \in J \text{ there is a } j \in J \text{ such that } j \geq k \text{ and } A_j \cap U \neq \emptyset\},$$

and

$$\lim I(A_j) = \{x \in X: \text{for every neighborhood } U \text{ of } x \text{ there is a } k \in J \text{ such that } j \geq k \text{ implies } A_j \cap U \neq \emptyset\}.$$ 

The net $\{A_j\}$ is termed topologically convergent if $\lim S(A_j) = \lim I(A_j)$. If $\{A_j\}$ is topologically convergent, the set $\lim S(A_j)$ is called the limit of $\{A_j\}$. We note that $\lim S(A_j)$ may be the empty set.

If $X$ is second countable, it is known that every sequence of subsets of $X$ contains a subsequence which is topologically convergent [3, Theorem VIII, p. 246]. The purpose of this section is to prove the following theorem.

**Theorem.** If $X$ is first countable, every net of subsets of $X$ contains a topologically convergent subnet.

**Proof.** We will first show that if $\{B_k\}$ is any net of subsets of $X$, then there is a subnet $\{B_h: h \in H\}$ of $\{B_k\}$ such that

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To prove the above, put
\[ \psi_k(x) = 1 \quad \text{if } x \text{ is in } B_k, \]
\[ = 0 \quad \text{otherwise}. \]

Let \( P = \{0, 1\}\). Being a product of compact spaces, \( P \) is compact. Hence there is a subnet \( \{\psi_h : h \in H\} \) of \( \{\psi_k\} \) which is convergent in the product topology on \( P \). Let
\[ B_h = \{ x \in X : \psi_h(x) = 1 \}. \]

By direct computation one can verify that \( \{B_h : h \in H\} \) is a subnet of \( \{B_k\} \) satisfying the property stated above.

Now let \( \{A_j : j \in J_0\} \) be a net of subsets of \( X \). For each \( x \) in \( X \), let \( \{V(x, k) : k = 1, 2, \ldots\} \) be a countable base for the neighborhood system for \( x \). We may assume that \( V(x, k) \) contains \( V(x, k+1) \). Now for \( j \) in \( J_0 \) and \( k = 1, 2, \ldots \) let
\[ B(k, j, 0) = \{ x \in X : V(x, k) \cap A_j \neq \emptyset \}. \]

From the first part of the proof there is a subnet of \( \{B(1, j, 0) : j \in J_0\} \) say \( \{B(1, j, 1) : j \in J_1\} \) such that
\[ \bigcup_{j \in J_1} \bigcap_{h \geq 0} B(1, j, 1) = \bigcap_{j \in J_1} \bigcup_{h \geq 0} B(1, j, 1). \]

Having constructed the net \( \{B(i-1, j, i-1) : j \in J_{i-1}\} \), we consider the net of subsets \( \{B(i, j, i-1) : j \in J_{i-1}\} \). As above, this net admits a subnet \( \{B(i, j, i) : j \in J_i\} \) such that
\[ \bigcup_{j \in J_i} \bigcap_{h \geq 0} B(i, j, i) = \bigcap_{j \in J_i} \bigcup_{h \geq 0} B(i, j, i). \]

Let \( D = \times \{J_i : i = 1, 2, \ldots\} \) have the product ordering. For \( d \) in \( D \), let \( d(i) \) denote the \( i \)th coordinate of \( d \). Let \( I \) denote the positive integers, and let \( I \times D \) have the product ordering. For \( s = (i, d) \) in \( S \), let \( R(s) = (i, d(i), i) \). Let \( B = \cap_{i=1}^{\infty} \bigcup_{d \in J_i} \bigcap_{h \geq 0} B(i, h, i) \). We intend to show that the net \( \{A_{R(s)} : s \in S\} \) converges topologically to \( B \). Since this net is a subnet of \( \{A_j : j \in J_0\} \), this will complete the proof of the theorem.

Now suppose \( x \) is in \( B \). We will show that every neighborhood of \( x \) eventually meets \( A_{R(s)} \). From the definitions of \( B \) and \( B(i, j, i) \) we have that, for every positive integer \( k \),
\[ \{V(x, k) \cap A_{<k,j,k>} : j \in J_k\} \]
eventually consists of nonempty sets. Now \( \{A_{R(i, d)}: i > k, d \in D\} \) is a subnet of \( \{A_{(k, j, h)}: j \in J_k\} \). Hence \( \{V(x, k) \cap A_{R(s)}: s \in S\} \) eventually consists of nonempty sets as we intended to show.

If \( x \) is not in \( B \), there is a \( k \) such that \( x \) is not in \( \bigcup_{\substack{g \in J_k \\cap \\{h \in J_k: h \leq g\}}} B(k, h, k) = \bigcup_{g \in J_k} B(k, h, k) \).

Hence \( \{V(x, k) \cap A_{(k, j, h)}: j \in J_k\} \) eventually consists of empty sets. Since \( \{A_{R(i, d)}: i > k, d \in D\} \) is a subnet of \( \{A_{(k, j, h)}: j \in J_k\} \), we have that \( \{V(x, k) \cap A_{R(s)}: s \in S\} \) eventually consists of empty sets.

We have shown that if \( x \) is not in \( B \), then \( x \) has a neighborhood which eventually has empty intersection with \( A_{R(s)} \). We conclude that \( \{A_{R(s)}\} \) converges to \( B \). This completes our proof.

We do not know, if under the hypothesis of the theorem, every sequence of subsets of \( X \) contains a convergent subsequence.

III. Finite topology. Let \( X \) be a Hausdorff space, and let \( Q \) denote the collection of nonempty compact subsets of \( X \). For a finite collection of open sets in \( X \) \( \{U_i: i = 1, 2, \ldots, n\} \), we define

\[
[U_1, U_2, \ldots, U_n] = \left\{ K \in Q: K \cap U_i \neq \emptyset, i = 1, 2, \ldots, n, \text{ and } K \subseteq \bigcup_{i=1}^{n} U_i \right\}.
\]

The collection of sets

\[
\{[U_1, U_2, \ldots, U_n]: U_i \text{ open in } X \text{ for } i = 1, 2, \ldots, n\}
\]

is a base for a topology on \( Q \) called the finite (Vietoris) topology. We henceforth assume that \( Q \) has the finite topology.

It is known [4, §4] that \( Q \) is completely regular, metrizable, compact, separable, or second countable if and only if \( X \) is respectively completely regular, metrizable, compact, separable, or second countable. In this section we characterize the spaces \( X \) for which \( Q \) is first countable.

A subset \( P \) of \( X \) is said to have a countable exterior base if there is a countable family of open sets in \( X \) whose intersection is \( P \) and such that every open set containing \( P \) also contains some member of the countable family. If \( X \) is compact, a closed set \( P \) has a countable exterior base if and only if it is a \( G_\delta \) set.

**Theorem.** In order that \( Q \) is first countable, it is necessary and sufficient that the following conditions are satisfied:

1. Every compact subset of \( X \) is separable.
2. Every compact subset of \( X \) has a countable exterior base.
Proof. Suppose that \( \{ U(i, j) : i = 1, 2, \cdots, n_j \} : j = 1, 2, \cdots \) is a countable base in the finite topology for the neighborhood system of an element \( P \) in \( Q \). Let \( b(i, j) \) be a point in the intersection of \( U(i, j) \) and \( P \). We will show that \( \{ b(i, j) : i = 1, 2, \cdots, n_j ; j = 1, 2, \cdots \} \) is dense in \( P \). Let \( V \) be an open set in \( X \) which has nonempty intersection with \( P \). Since \( [X, V] \) is a neighborhood of \( P \), there is a \( k \) such that \( \{ U(i, k) : i = 1, 2, \cdots, n_k \} \) is contained in \( [X, V] \). If for each \( i = 1, 2, \cdots, n_k \), there is a point \( y_i \) in \( U[i, k] - V \), then \( \{ y_1, y_2, \cdots, y_{n_k} \} \) would be a compact set contained in \( \{ U(i, k) : i = 1, 2, \cdots, n_k \} \) but not in \( [X, V] \). This, however, is not possible. Hence \( V \) contains one of the sets \( U(i, k) : i = 1, 2, \cdots, n_k \), and \( V \) contains one of the points \( b(i, j) \). Hence \( P \) is separable.

If \( W \) is an open set containing \( P \), then \( [W] \) is a neighborhood of \( P \). For some \( k \) \( \{ U(i, k) : i = 1, 2, \cdots, n_k \} \) is contained in \( [W] \) and \( \bigcup_{i=1}^{n_k} U(i, k) \) is contained in \( W \). It follows that \( P \) has a countable exterior base.

To prove the converse let \( P \) be a compact set in \( X \), and assume that \( X \) satisfies properties (1) and (2) in the statement of the theorem. Let \( \{ U(i) : i = 1, 2, \cdots \} \) be a countable family of open sets which form a countable exterior base for \( P \). Let \( \{ x(i) : i = 1, 2, \cdots \} \) be a countable dense subset of \( P \). For \( i = 1, 2, \cdots \) let \( \{ V(i, j) : j = 1, 2, \cdots \} \) be a countable base for the neighborhood system in \( X \) of the point \( x(i) \). Let \( W = \bigcup \{ V(i, j) : i = 1, 2, \cdots ; j = 1, 2, \cdots \} \) Now \( \{ [U(i), W_1, W_2, \cdots, W_n] : W_1, W_2, \cdots, W_n \text{ in } W \text{ and } i = 1, 2, \cdots \} \) is a countable base in the finite topology for the neighborhood system for \( P \). This completes the proof.

In the following corollaries let \( X \) be a compact Hausdorff space.

Corollary. If \( Q \) is first countable then \( X \) is a first countable, separable, perfectly normal space (see [2, p. 134]).

Corollary. If \( Q \) is first countable, \( Q \) is separable.

Example. Let \( X = [0, 1] \times [0, 1] \) be ordered lexicographically and given the order topology. Then \( X \) is first countable, Hausdorff, and compact but not separable [2, p. 164].

Example. Let \( X \) be as in the example above. Let \( P = \{ (r, i) \text{ in } X : i = 0 \text{ or } i = 1 \text{ and } 0 \leq r \leq 1 \} \).

Now \( P \) is compact and separable, but \( P \) does not have a countable exterior base. For if \( V \) is an open set containing \( P \) there can exist at most finitely many real values \( x_i \) such that \( 0 < x_i < 1 \) and there is a \( y_i \).
for which $0 < y_i < 1$ and $(x_i, y_i)$ is not in $V$. (If there did exist an infinite number of such points $x_i$ we could construct a sequence of points not in $V$ that converged to a point in $P$. This of course is not possible.) Therefore any countable intersection of open sets in $X$ which contains $P$ must also contain points not in $P$. Hence $P$ does not have a countable exterior base.

**References**


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