SUBSETS OF FIRST COUNTABLE SPACES

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I. **Summary.** Let X be a topological space. If X is second countable, it is known [**3**, Theorem VIII, p. 246] that any sequence of subsets of X contains a subsequence which is topologically convergent. In §II of this note we show that if X is first countable, any net of subsets of X contains a subnet which is topologically convergent. (See §II for definitions.)

Let Q denote the space of compact nonempty subsets of a Hausdorff space X, topologized with the finite topology. In §III we characterize the spaces X for which Q is first countable. It previously had been thought that the first countability of X was sufficient to imply first countability of Q [**4**, Proposition 4.5.3, p. 162]. Appropriate counter examples are found in §III.

II. **Topological convergence of sets.** Let $\{A_j: j \text{ in } J\}$ be a net of subsets (not necessarily closed or nonempty) of a topological space X.

$$\lim S(A_j) = \{x \text{ in } X\text{: for every neighborhood } U \text{ of } x \text{ and any } k \text{ in } J \text{ there is a } j \text{ in } J \text{ such that } j \geqq k \text{ and } A_j \cap U \neq \emptyset\},$$

and

$$\lim I(A_j) = \{x \text{ in } X\text{: for every neighborhood } U \text{ of } x \text{ there is a } k \text{ in } J \text{ such that } j \geqq k \text{ implies } A_j \cap U \neq \emptyset\}.$$

The net $\{A_j\}$ is termed *topologically convergent* if $\lim S(A_j) = \lim I(A_j)$. If $\{A_j\}$ is topologically convergent, the set $\lim S(A_j)$ is called the *limit* of $\{A_j\}$. We note that $\lim S(A_j)$ may be the empty set.

If X is second countable, it is known that every sequence of subsets of X contains a subsequence which is topologically convergent [**3**, Theorem VIII, p. 246]. The purpose of this section is to prove the following theorem.

THEOREM. *If X is first countable, every net of subsets of X contains a topologically convergent subnet.*

PROOF. We will first show that if $\{B_k\}$ is any net of subsets of X, then there is a subnet $\{B_h: h \text{ in } H\}$ of $\{B_k\}$ such that

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$$\bigcup_{g \text{ in } H} \bigcap_{h \geq g} B_h = \bigcap_{g \text{ in } H} \bigcup_{h \geq g} B_h.$$

To prove the above, put

$$\begin{aligned} \psi_k(x) &= 1 \qquad \text{if } x \text{ is in } B_k, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let $P = \{0, 1\}^E$. Being a product of compact spaces, P is compact. Hence there is a subnet $\{\psi_h: h \text{ in } H\}$ of $\{\psi_k\}$ which is convergent in the product topology on P. Let

$$B_h = \{x \text{ in } X: \psi_h(x) = 1\}.$$

By direct computation one can verify that $\{B_h: h \text{ in } H\}$ is a subnet of $\{B_k\}$ satisfying the property stated above.

Now let $\{A_j: j \text{ in } J_0\}$ be a net of subsets of X. For each x in X, let $\{V(x, k): k = 1, 2, \cdots\}$ be a countable base for the neighborhood system for x. We may assume that $V(x, k)$ contains $V(x, k+1)$. Now for j in J_0 and $k = 1, 2, \cdots$ let

$$B(k, j, 0) = \{x \text{ in } X: V(x, k) \cap A_j \neq \emptyset\}.$$

From the first part of the proof there is a subnet of $\{B(1, j, 0): j \text{ in } J_0\}$ say $\{B(1, j, 1): j \text{ in } J_1\}$ such that

$$\bigcup_{g \text{ in } J_1} \bigcap_{h \geq g} B(1, j, 1) = \bigcap_{g \text{ in } J_1} \bigcup_{h \geq g} B(1, j, 1).$$

Having constructed the net $\{B(i-1, j, i-1): j \text{ in } J_{i-1}\}$, we consider the net of subsets $\{B(i, j, i-1): j \text{ in } J_{i-1}\}$. As above, this net admits a subnet $\{B(i, j, i): j \text{ in } J_i\}$ such that

$$\bigcup_{g \text{ in } J_i} \bigcap_{h \geq g} B(i, j, i) = \bigcap_{g \text{ in } J_i} \bigcup_{h \geq g} B(i, j, i).$$

Let $D = \times\{J_i: i = 1, 2, \cdots\}$ have the product ordering. For d in D, let $d(i)$ denote the ith coordinate of d. Let I denote the positive integers, and let $I \times D$ have the product ordering. For $s = (i, d)$ in S, let $R(s) = (i, d(i), i)$. Let $B = \bigcap_{i=1}^{\infty} \bigcup_{g \text{ in } J_i} \bigcap_{h \geq g} B(i, h, i)$. We intend to show that the net $\{A_{R(s)}: s \text{ in } S\}$ converges topologically to B. Since this net is a subnet of $\{A_j: j \text{ in } J_0\}$, this will complete the proof of the theorem.

Now suppose x is in B. We will show that every neighborhood of x eventually meets $A_{R(s)}$. From the definitions of B and $B(i, j, i)$ we have that, for every positive integer k,

$$\{V(x, k) \cap A_{(k,j,k)}: j \text{ in } J_k\}$$

eventually consists of nonempty sets. Now $\{A_{R(i,d)}: i>k, d \text{ in } D\}$ is a subnet of $\{A_{(k,j,k)}: j \text{ in } J_k\}$. Hence $\{V(x, k)\cap A_{R(s)}: s \text{ in } S\}$ eventually consists of nonempty sets as we intended to show.

If x is not in B, there is a k such that x is not in

$$\bigcup_{g \text{ in } J_k} \bigcap_{h\geqq g} B(k, h, k) = \bigcap_{g \text{ in } J_k} \bigcup_{h\geqq g} B(k, h, k).$$

Hence $\{V(x, k)\cap A_{(k,j,k)}: j \text{ in } J_k\}$ eventually consists of empty sets. Since $\{A_{R(i,d)}: i>k, d \text{ in } D\}$ is a subnet of $\{A_{(k,j,k)}: j \text{ in } J_k\}$, we have that $\{V(x, k)\cap A_{R(s)}: s \text{ in } S\}$ eventually consists of empty sets.

We have shown that if x is not in B, then x has a neighborhood which eventually has empty intersection with $A_{R(s)}$. We conclude that $\{A_{R(s)}\}$ converges to B. This completes our proof.

We do not know, if under the hypothesis of the theorem, every sequence of subsets of X contains a convergent subsequence.

III. **Finite topology.** Let X be a Hausdorff space, and let Q denote the collection of nonempty compact subsets of X. For a finite collection of open sets in X $\{U_i: i=1, 2, \cdots, n\}$, we define

$$[U_1, U_2, \cdots, U_n] = \left\{K \text{ in } Q\colon K\cap U_i \neq \varnothing,\ i = 1, 2, \cdots, n, \text{ and } K\subseteq \bigcup_{i=1}^{n} U_i\right\}.$$

The collection of sets

$$\{[U_1, U_2, \cdots, U_n]\colon U_i \text{ open in } X \text{ for } i = 1, 2, \cdots, n\}$$

is a base for a topology on Q called the *finite* (*Vietoris*) *topology*. We henceforth assume that Q has the finite topology.

It is known [**4**, §4] that Q is completely regular, metrizable, compact, separable, or second countable if and only if X is respectively completely regular, metrizable, compact, separable, or second countable. In this section we characterize the spaces X for which Q is first countable.

A subset P of X is said to have a *countable exterior base* if there is a countable family of open sets in X whose intersection is P and such that every open set containing P also contains some member of the countable family. If X is compact, a closed set P has a countable exterior base if and only if it is a G_δ set.

THEOREM. *In order that Q is first countable, it is necessary and sufficient that the following conditions are satisfied*:

(1) *Every compact subset of X is separable.*

(2) *Every compact subset of X has a countable exterior base.*

PROOF. Suppose that $\{[U(i, j): i=1, 2, \cdots, n_j]: j=1, 2, \cdots\}$ is a countable base in the finite topology for the neighborhood system of an element P in Q. Let $b(i, j)$ be a point in the intersection of $U(i, j)$ and P. We will show that $\{b(i, j): i=1, 2, \cdots, n_j; j=1, 2, \cdots\}$ is dense in P. Let V be an open set in X which has nonempty intersection with P. Since $[X, V]$ is a neighborhood of P, there is a k such that $[U(i, k): i=1, 2, \cdots, n_k]$ is contained in $[X, V]$. If for each $i=1, 2, \cdots, n_k$, there is a point y_i in $U[i, k]-V$, then $\{y_1, y_2, \cdots, y_{n_k}\}$ would be a compact set contained in $[U(i, k): i=1, 2, \cdots, n_k]$ but not in $[X, V]$. This, however, is not possible. Hence V contains one of the sets $U(i, k): i=1, 2, \cdots, n_k$, and V contains one of the points $b(i, j)$. Hence P is separable.

If W is an open set containing P, then $[W]$ is a neighborhood of P. For some k $[U(i, k): i=1, 2, \cdots, n_k]$ is contained in $[W]$ and $\bigcup_{i=1}^{n_k} U(i, k)$ is contained in W. It follows that P has a countable exterior base.

To prove the converse let P be a compact set in X, and assume that X satisfies properties (1) and (2) in the statement of the theorem. Let $\{U(i): i=1, 2, \cdots\}$ be a countable family of open sets which form a countable exterior base for P. Let $\{x(i): i=1, 2, \cdots\}$ be a countable dense subset of P. For $i=1, 2, \cdots$ let $\{V(i, j): j=1, 2, \cdots\}$ be a countable base for the neighborhood system in X of the point $x(i)$. Let $W=\bigcup\{V(i, j): i=1, 2, \cdots; j=1, 2, \cdots\}$ Now

$$\{[U(i), W_1, W_2, \cdots, W_n]: W_1, W_2, \cdots, W_n \text{ in } W \text{ and } i = 1, 2, \cdots\}$$

is a countable base in the finite topology for the neighborhood system for P. This completes the proof.

In the following corollaries let X be a compact Hausdorff space.

COROLLARY. *If Q is first countable then X is a first countable, separable, perfectly normal space (see [2, p. 134]).*

COROLLARY. *If Q is first countable, Q is separable.*

EXAMPLE. Let $X=[0, 1]\times[0, 1]$ be ordered lexicographically and given the order topology. Then X is first countable, Hausdorff, and compact but not separable [2, p. 164].

EXAMPLE. Let X be as in the example above. Let

$$P = \{(r, i) \text{ in } X: i = 0 \text{ or } i = 1 \text{ and } 0 \leqq r \leqq 1\}.$$

Now P is compact and separable, but P does not have a countable exterior base. For if V is an open set containing P there can exist at most finitely many real values x_i such that $0<x_i<1$ and there is a y_i

for which $0 < y_i < 1$ and (x_i, y_i) is not in V. (If there did exist an infinite number of such points x_i we could construct a sequence of points not in V that converged to a point in P. This of course is not possible.) Therefore any countable intersection of open sets in X which contains P must also contain points not in P. Hence P does not have a countable exterior base.

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