1. Introduction and results. In [2] and [3], K. Nagami and the
author introduced functions $d_2$ and $d_3$ from the class of all $(X, \rho)$ into
the nonnegative integers, where $X$ is a nonnull metrizable topological
space and $\rho$ is a metric for $X$, consistent with the topology of $X$.
Formal definitions are given in [3], and are condensed as follows.

Definition. $d_2(X, \rho)$ is the smallest integer $n$ such that for every
set of $n + 1$ pairs $C_1, C'_1; C_2, C'_2; \cdots; C_{n+1}, C'_{n+1}$ of closed subsets of $X$
with $\rho(C_i, C'_i) > 0$ for $i = 1, 2, \ldots, n + 1$ there exist closed sets
$B_1, B_2, \ldots, B_{n+1}$ such that (i) $B_i$ separates $X$ between $C_i$ and $C'_i$
and (ii) $\bigcap_{i=1}^{n+1} B_i = \emptyset$.

Definition. $d_3(X, \rho)$ is the smallest integer $n$ such that given any
positive integer $m$ and $m$ pairs $C_1, C'_1; C_2, C'_2; \cdots; C_m, C'_m$ of closed
subsets of $X$ with $\rho(C_i, C'_i) > 0$ for $i = 1, 2, \ldots, m$ then there exist
closed sets $B_1, B_2, \ldots, B_m$ such that (i) $B_i$ separates $X$ between $C_i$
and $C'_i$ and (ii) no $n + 1$ of the sets $B_1, B_2, \ldots, B_m$ have a point in
common (order $\{B_i: i = 1, 2, \ldots, m\} \leq n$).

We also consider metric dimension of $(X, \rho)$, denoted $\mu \dim(X, \rho)$,
and defined as the smallest integer $n$ such that for all $\epsilon > 0$ the set of
all $\epsilon$-balls has a refining cover of order $n + 1$ or less.

The following results are in [3]:
(i) for all $(X, \rho)$, $d_2(X, \rho) \leq d_3(X, \rho) \leq \mu \dim(X, \rho) \leq \dim X$,
(ii) for totally bounded $(X, \rho)$, $d_3(X, \rho) = \mu \dim(X, \rho)$, and
(iii) for all $n > 2$ there exists a space $(X_n, \rho)$ with $d_2(X_n, \rho) < d_3(X_n, \rho)$
$\leq \mu \dim(X_n, \rho) < \dim X = n$.

No example is known having $d_3$ strictly less than $\mu \dim$.

The main result of the present paper is stated in Theorem 1. Using
this theorem we can prove Theorem 2, which extends to the function
$d_3$ (provided $X$ is separable) a result proved in [4] for the
function $\mu \dim$.

Theorem 1. Let $(X, \rho)$ be a separable metric space. Then there exists
a homeomorphism $h$ of $X$ into a subset of $I^\omega$ (the Hilbert cube) such that,
letting $\sigma$ denote the $I^\omega$ metric
(i) $d_2(X, \rho) = d_2(h(X), \sigma)$ and
(ii) $d_3(X, \rho) = d_3(h(X), \sigma)$.
Thus for any separable $(X, \rho)$ there is a topologically equivalent totally
bounded metric $\sigma$ which preserves $d_2$ and $d_3$.

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Theorem 2. For a separable metric space \( (X, \rho) \), suppose \( d_3(X, \rho) = r \), \( \dim X = n \), and \( r < n \). Then for every integer \( k \) \( (r \leq k \leq n) \) there exists a metric \( \rho_k \) for \( X \) such that

(i) \( \rho_k \) is topologically equivalent to \( \rho \), and
(ii) \( d_3(X, \rho_k) = k \).

Unsolved Problem. In the statement of Theorem 2, replace \( d_3 \) by \( d_2 \). Is the resulting statement true?

2. Proof of Theorem 1. Define \( r \) and \( s \) as follows: \( d_3(X, \rho) = r \), \( d_3(X, \rho) = s \). Then since \( d_2(X, \rho) \geq r \) there exist \( 2r \) closed sets \( C_1, C'_1 \); \( C_2, C'_2 \); \ldots; \( C_r, C'_r \) such that (i) \( \rho(C_i, C'_i) > 0 \) for \( 1 \leq i \leq r \) and (ii) if for each \( i \) the closed set \( B_i \) separates \( X \) between \( C_i \) and \( C'_i \) then \( \cap_{i=r} B_i \neq \emptyset \). Similarly, since \( d_3(X, \rho) \geq s \), there is an integer \( m \) and \( 2m \) closed sets \( C_{r+1}, C'_{r+1} \); \ldots; \( C_{r+m}, C'_{r+m} \), such that (i) \( \rho(C_i, C'_i) > 0 \) and (ii) if for each \( i \) \( (r \leq i \leq r+m) \), the closed set \( B_i \) separates \( X \) between \( C_i \) and \( C'_i \) then \( \left\{ B_i : r+1 \leq i \leq r+m \right\} \geq s \).

For each positive integer \( i \) we define a function \( f_i : X \rightarrow \mathbb{I}^n \), and define \( h : X \rightarrow I^\omega \) by the formula

\[
 h(x) = (f_1(x), f_2(x), \ldots) \in I^\omega.
\]

Let \( \{\rho_i : i \geq r+m\} \) be a countable dense subset of \( X \) and make the following definitions:

\[
 f_i(x) = \frac{\rho(x, C_i)}{i(\rho(x, C_i) + \rho(x, C'_i))} \quad (i \leq r + m);
\]

\[
 f_i(x) = \frac{\rho(x, \rho_i)}{i(1 + \rho(x, \rho_i))} \quad (i > r + m).
\]

Letting \( \sigma \) be the usual metric in \( I^\omega \) we have, for \( x, y \in X \),

\[
 \sigma(h(x), h(y)) = \left( \sum_{i=1}^{\infty} (f_i(x) - f_i(y))^2 \right)^{1/2}.
\]

Note that for

\[
 x \in C_i, \quad y \in C'_i, \quad f_i(y) - f_i(x) = f_i(y) = 1/i
\]

so that

\[
 \sigma(h(C_i), h(C'_i)) > 0.
\]

2.1. Lemma. Let \( \delta > 0 \) be the smaller of 1 and the minimum \( \rho(C_i, C'_i)/4 \) for \( 1 \leq i \leq r + m \). Let \( \epsilon \) be given such that \( 0 < \epsilon < \delta \), and let \( \eta = \epsilon \delta / 2 \). Then if \( x, y \in X \) and \( \rho(x, y) < \eta \) it follows that \( |f_i(x) - f_i(y)| < \epsilon / 2i \) for all \( i \), and \( \sigma(h(x), h(y)) < \epsilon \).
Proof. Fix $x$ and $y$ such that $\rho(x, y) < \eta$. Fix $i \leq r+m$. To simplify notation, introduce $A$, $B$, $\alpha$, and $\beta$ by the definitions

$$\rho(x, C_i) = A, \quad \rho(x, C'_i) = B, \quad \rho(y, C_i) = A + \alpha, \quad \rho(y, C'_i) = B + \beta.$$ 

Then

$$|\alpha| < \eta, \quad |\beta| < \eta, \quad A + B + \alpha + \beta > 4\delta - 2\eta > 2\delta,$$

and (see (2))

$$i \left| f_i(x) - f_i(y) \right| = \left\| \frac{A}{A + B} - \frac{A + \alpha}{A + B + \alpha + \beta} \right\| \leq \frac{A}{A + B} \frac{|\beta|}{2\delta} + \frac{B}{A + B} \frac{|\alpha|}{2\delta} < \frac{\epsilon \delta}{2\delta} = \frac{\epsilon}{2}.$$

Thus for $i \leq r+m$ we have $|f_i(x) - f_i(y)| < \epsilon/2i$. If $i > r+m$, from (3) it is trivial that $i |f_i(x) - f_i(y)| \leq \rho(x, y) < \epsilon/2$. Thus for all $i$, $|f_i(x) - f_i(y)| < \epsilon/2i$, and, using the fact that $\sum_{i=1}^{n} 1/i^2 = \pi^2/6 < 4$, we have $\sigma(h(x), h(y)) < \epsilon$, from (4). This completes the proof of the lemma.

2.2. A sufficient condition that $h : X \to h(X)$ be a homeomorphism is that for $x \in X$, $M \subset X$, (i) if $\rho(x, M) = 0$ then $\sigma(h(x), h(M)) = 0$, and (ii) if $\rho(x, M) > 0$ then $\sigma(h(x), h(M)) > 0$. Statement (i) follows trivially from Lemma 2.1. To prove (ii) suppose $\rho(x, M) = d > 0$, and fix $i (i > r+m)$ so that $\rho(x, p_i) < d/4$. Then for all $y \in M$ we have $\rho(y, p_i) > 3d/4$, and

$$f_i(y) - f_i(x) > \frac{3d/4}{i(1 + 3d/4)} - \frac{d/4}{i(1 + d/4)} = \epsilon > 0,$$

with $\epsilon$ independent of $y$.

Thus

$$\sigma(h(x), h(M)) \geq \inf \{(f_i(y) - f_i(x)) : y \in M\} \geq \epsilon.$$ 

2.3. Assertion. If $C, C'$ are disjoint closed subsets of $X$ and $\rho(C, C') = 0$, then $\sigma(h(C), h(C')) = 0$.

Proof. Let $\delta$, $\epsilon$, and $\eta$ be given as in the hypothesis of Lemma 2.1. Fix $x \in C$, $y \in C'$ so that $\rho(x, y) < \eta$. Then $\sigma(h(C), h(C')) \leq \sigma(h(x), h(y)) \leq \epsilon$, by Lemma 2.1. Thus $\sigma(h(C), h(C')) = 0$.

2.4. Conclusion of Proof of Theorem 1. In view of 2.3, it is evident from definitions that $d_2(h(X), \sigma) \leq r$ and $d_3(h(X), \sigma) \leq s$, because in $(h(X), \sigma)$ there is no "new" pair $C, C'$ at positive distance. On the other hand, the $r+m$ pairs $C_1, C'_1; \ldots; C_{r+m}, C'_{r+m}$, which
guarantee that \( d_2(X, \rho) \geq r, \ d_3(X, \rho) \geq s \) remain at positive distance in \( (h(X), \sigma) \) so \( d_2(h(X), \sigma) \geq r, \ d_3(h(X), \sigma) \geq s. \)

3. Proof of Theorem 2. We are given a separable metric space \((X, \rho)\) with \( d_3(X, \rho) = r < n = \dim X \). From Theorem 1 there is a topologically equivalent metric \( \sigma \) for \( X \) such that (i) \( d_3(X, \sigma) = d_3(X, \rho) \) and (ii) \((X, \sigma)\) is totally bounded. Thus [3, Theorem 5] \( d_3(X, \sigma) = \mu \dim(X, \sigma) = r \). Now in [4], in the proof of the main theorem, a finite number of continuous functions \( f_1, f_2, \cdots, f_t \) are defined, \( f_i: X \to [0, 1] \), and metrics \( \sigma_1, \sigma_2, \cdots, \sigma_t \) for \( X \) are defined by the formula

\[
\sigma_i(x, y) = \sigma_{i-1}(x, y) + |f_i(x) - f_i(y)|,
\]

where \( \sigma_0 = \sigma \). These have the property that \( \mu \dim(X, \sigma_i) \leq \mu \dim(X, \sigma_{i+1}) \leq \mu \dim(X, \sigma_i) + 1 \), and \( \mu \dim(X, \sigma_i) = n \). Thus for any \( k \ (r < k \leq n) \) there exists \( i(k) \) such that \( \mu \dim(X, \sigma_{i(k)}) = k \). But in the present case, with \((X, \sigma_0)\) totally bounded, every \((X, \sigma_i)\) is totally bounded (see Hurewicz [1, p. 200]) so by [3] we have \( d_3(X, \sigma_{i(k)}) = k \), and the proof is complete.

References


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