REALIZABILITY OF METRIC-DEPENDENT DIMENSIONS[1]

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1. **Introduction and results.** In [2] and [3], K. Nagami and the author introduced functions d_2 and d_3 from the class of all (X, ρ) into the nonnegative integers, where X is a nonnull metrizable topological space and ρ is a metric for X, consistent with the topology of X. Formal definitions are given in [3], and are condensed as follows.

DEFINITION. $d_2(X, \rho)$ is the smallest integer n such that for every set of $n+1$ pairs C_1, C_1'; C_2, C_2'; $\cdots$; C_{n+1}, C_{n+1}' of closed subsets of X with $\rho(C_i, C_i')>0$ for $i=1, 2, \cdots, n+1$ there exist closed sets $B_1, B_2, \cdots, B_{n+1}$ such that (i) B_i separates X between C_i and C_i' and (ii) $\bigcap_{i=1}^{n+1} B_i=\emptyset$.

DEFINITION. $d_3(X, \rho)$ is the smallest integer n such that given any positive integer m and m pairs C_1, C_1'; C_2, C_2'; $\cdots$; C_m, C_m' of closed subsets of X with $\rho(C_i, C_i')>0$ for $i=1, 2, \cdots, m$ then there exist closed sets $B_1, B_2, \cdots, B_m$ such that (i) B_i separates X between C_i and C_i' and (ii) no $n+1$ of the sets $B_1, B_2, \cdots, B_m$ have a point in common (order $\{B_i: i=1, 2, \cdots, m\} \leqq n$).

We also consider metric dimension of (X, ρ), denoted $\mu \dim(X, \rho)$, and defined as the smallest integer n such that for all $\epsilon>0$ the set of all ϵ-balls has a refining cover of order $n+1$ or less.

The following results are in [3]:

(i) for all (X, ρ), $d_2(X, \rho) \leqq d_3(X, \rho) \leqq \mu \dim(X, \rho) \leqq \dim X$,

(ii) for totally bounded (X, ρ), $d_3(X, \rho)=\mu \dim(X, \rho)$, and

(iii) for all $n>2$ there exists a space (X_n, ρ) with $d_2(X_n, \rho)<d_3(X_n, \rho) \leqq \mu \dim(X_n, \rho)<\dim X=n$.

No example is known having d_3 strictly less than $\mu \dim$.

The main result of the present paper is stated in Theorem 1. Using this theorem we can prove Theorem 2, which extends to the function d_3 (provided X is separable) a result proved in [4] for the function $\mu \dim$.

THEOREM 1. *Let (X, ρ) be a separable metric space. Then there exists a homeomorphism h of X into a subset of I^ω (the Hilbert cube) such that, letting σ denote the I^ω metric*

(i) $d_2(X, \rho)=d_2(h(X), \sigma)$ *and*

(ii) $d_3(X, \rho)=d_3(h(X), \sigma)$.

Thus for any separable (X, ρ) there is a topologically equivalent totally bounded metric σ which preserves d_2 and d_3.

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THEOREM 2. *For a separable metric space* (X, ρ), *suppose* $d_3(X, \rho) = r$, $\dim X = n$, *and* $r < n$. *Then for every integer* k $(r \leqq k \leqq n)$ *there exists a metric* ρ_k *for* X *such that*

(i) ρ_k *is topologically equivalent to* ρ, *and*

(ii) $d_3(X, \rho_k) = k$.

UNSOLVED PROBLEM. In the statement of Theorem 2, replace d_3 by d_2. Is the resulting statement true?

2. **Proof of Theorem 1.** Define r and s as follows: $d_2(X, \rho) = r$, $d_3(X, \rho) = s$. Then since $d_2(X, \rho) \geqq r$ there exist $2r$ closed sets C_1, C_1'; C_2, C_2'; $\cdots$; C_r, C_r' such that (i) $\rho(C_i, C_i') > 0$ for $1 \leqq i \leqq r$ and (ii) if for each i the closed set B_i separates X between C_i and C_i' then $\bigcap_{i=1}^r B_i \neq \emptyset$. Similarly, since $d_3(X, \rho) \geqq s$, there is an integer m and $2m$ closed sets C_{r+1}, C_{r+1}'; $\cdots$; C_{r+m}, C_{r+m}', such that (i) $\rho(C_i, C_i') > 0$ and (ii) if for each i $(r < i \leqq r+m)$, the closed set B_i separates X between C_i and C_i' then $\operatorname{ord}\{B_i: r+1 \leqq i \leqq r+m\} \geqq s$.

For each positive integer i we define a function $f_i: X \to [0, 1/i]$, and define $h: X \to I^\omega$ by the formula

$$h(x) = (f_1(x), f_2(x), \cdots) \in I^\omega. \tag{1}$$

Let $\{p_i: i > r+m\}$ be a countable dense subset of X and make the following definitions:

$$f_i(x) = \frac{\rho(x, C_i)}{i(\rho(x, C_i) + \rho(x, C_i'))} \qquad (i \leqq r+m); \tag{2}$$

$$f_i(x) = \frac{\rho(x, p_i)}{i(1 + \rho(x, p_i))} \qquad (i > r+m). \tag{3}$$

Letting σ be the usual metric in I^ω we have, for $x, y \in X$,

$$\sigma(h(x), h(y)) = \left(\sum_{i=1}^{\infty} (f_i(x) - f_i(y))^2 \right)^{1/2}. \tag{4}$$

Note that for

$$x \in C_i, \qquad y \in C_i', \qquad f_i(y) - f_i(x) = f_i(y) = 1/i$$

so that

$$\sigma(h(C_i), h(C_i')) > 0.$$

2.1. LEMMA. *Let* $\delta > 0$ *be the smaller of* 1 *and the minimum* $\rho(C_i, C_i')/4$ *for* $1 \leqq i \leqq r+m$. *Let* ϵ *be given such that* $0 < \epsilon < \delta$, *and let* $\eta = \epsilon\delta/2$. *Then if* $x, y \in X$ *and* $\rho(x, y) < \eta$ *it follows that* $|f_i(x) - f_i(y)| < \epsilon/2i$ *for all* i, *and* $\sigma(h(x), h(y)) < \epsilon$.

PROOF. Fix x and y such that $\rho(x, y) < \eta$. Fix $i \leqq r+m$. To simplify notation, introduce A, B, α, and β by the definitions

$$\rho(x, C_i) = A, \quad \rho(x, C_i') = B, \quad \rho(y, C_i) = A + \alpha, \quad \rho(y, C_i') = B + \beta.$$

Then

$$|\alpha| < \eta, \qquad |\beta| < \eta, \qquad A + B + \alpha + \beta > 4\delta - 2\eta > 2\delta,$$

and (see (2))

$$\begin{aligned} i\,|f_i(x) - f_i(y)| &= \left| \frac{A}{A+B} - \frac{A+\alpha}{A+B+\alpha+\beta} \right| \\ &\leqq \frac{A}{A+B}\,\frac{|\beta|}{2\delta} + \frac{B}{A+B}\,\frac{|\alpha|}{2\delta} < \frac{\epsilon\delta}{2\delta} = \frac{\epsilon}{2}. \end{aligned}$$

Thus for $i \leqq r+m$ we have $|f_i(x) - f_i(y)| < \epsilon/2i$. If $i > r+m$, from (3) it is trivial that $i|f_i(x) - f_i(y)| \leqq \rho(x, y) < \epsilon/2$. Thus for all i, $|f_i(x) - f_i(y)| < \epsilon/2i$, and, using the fact that $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6 < 4$, we have $\sigma(h(x), h(y)) < \epsilon$, from (4). This completes the proof of the lemma.

2.2. A sufficient condition that $h: X \to h(X)$ be a homeomorphism is that for $x \in X$, $M \subset X$, (i) if $\rho(x, M) = 0$ then $\sigma(h(x), h(M)) = 0$, and (ii) if $\rho(x, M) > 0$ then $\sigma(h(x), h(M)) > 0$. Statement (i) follows trivially from Lemma 2.1. To prove (ii) suppose $\rho(x, M) = d > 0$, and fix i $(i > r+m)$ so that $\rho(x, p_i) < d/4$. Then for all $y \in M$ we have $\rho(y, p_i) > 3d/4$, and

$$f_i(y) - f_i(x) > \frac{3d/4}{i(1 + 3d/4)} - \frac{d/4}{i(1 + d/4)} = \epsilon > 0,$$

with ϵ independent of y.

Thus

$$\sigma(h(x), h(M)) \geqq \inf\{(f_i(y) - f_i(x)) : y \in M\} \geqq \epsilon.$$

2.3. ASSERTION. If C, C' are disjoint closed subsets of X and $\rho(C, C') = 0$, then $\sigma(h(C), h(C')) = 0$.

PROOF. Let δ, ϵ, and η be given as in the hypothesis of Lemma 2.1. Fix $x \in C$, $y \in C'$ so that $\rho(x, y) < \eta$. Then $\sigma(h(C), h(C')) \leqq \sigma(h(x), h(y)) \leqq \epsilon$, by Lemma 2.1. Thus $\sigma(h(C), h(C')) = 0$.

2.4. CONCLUSION OF PROOF OF THEOREM 1. In view of 2.3, it is evident from definitions that $d_2(h(X), \sigma) \leqq r$ and $d_3(h(X, \sigma)) \leqq s$, because in $(h(X), \sigma)$ there is no "new" pair C, C' at positive distance. On the other hand, the $r+m$ pairs $C_1, C_1'; \cdots; C_{r+m}, C_{r+m}'$, which guarantee that $d_2(X, \rho) \geqq r$, $d_3(X, \rho) \geqq s$ remain at positive distance in $(h(X), \sigma)$ so $d_2(h(X), \sigma) \geqq r$, $d_3(h(X), \sigma) \geqq s$.

3. **Proof of Theorem 2.** We are given a separable metric space (X, ρ) with $d_3(X, \rho) = r < n = \dim X$. From Theorem 1 there is a topologically equivalent metric σ for X such that (i) $d_3(X, \sigma) = d_3(X, \rho)$ and (ii) (X, σ) is totally bounded. Thus [**3**, Theorem 5] $d_3(X, \sigma) = \mu \dim(X, \sigma) = r$. Now in [**4**], in the proof of the main theorem, a finite number of continuous functions $f_1, f_2, \cdots, f_t$ are defined, $f_i: X \to [0, 1]$, and metrics $\sigma_1, \sigma_2, \cdots, \sigma_t$ for X are defined by the formula

$$\sigma_i(x, y) = \sigma_{i-1}(x, y) + |f_i(x) - f_i(y)|,$$

where $\sigma_0 = \sigma$. These have the property that $\mu \dim(X, \sigma_i) \leqq \mu \dim(X, \sigma_{i+1}) \leqq \mu \dim(X, \sigma_i) + 1$, and $\mu \dim(X, \sigma_t) = n$. Thus for any k $(r < k \leqq n)$ there exists $i(k)$ such that $\mu \dim(X, \sigma_{i(k)}) = k$. But in the present case, with (X, σ_0) totally bounded, every (X, σ_i) is totally bounded (see Hurewicz [**1**, p. 200]) so by [**3**] we have $d_3(X, \sigma_{i(k)}) = k$, and the proof is complete.

References

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