CHARACTERIZATIONS OF METRIC-DEPENDENT DIMENSION FUNCTIONS[1]

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1. **Introduction.** Let (X, ρ) be a metric space, let $\dim(X)$ be the covering dimension of X, and let $d_0(X, \rho)$ be the metric dimension of X. Let d_2 and d_3 denote the metric-dependent dimension functions for metric spaces introduced by Nagami and Roberts [7], and let d_5 be the metric-dependent dimension function defined by Hodel [2]. A summary of the relationships among these dimension functions for (X, ρ) is the following:

$$d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_5(X, \rho) \leqq d_0(X, \rho) \leqq \dim(X) \leqq 2d_3(X, \rho).$$

In 1957, V. I. Egorov [1] characterized the dimension function d_0 in terms of Lebesgue covers as follows:

THEOREM. *Let (X, ρ) be a metric space. Then $d_0(X, \rho) \leqq n$ if and only if every Lebesgue cover of X has an open refinement of order $\leqq n+1$.*

In 1966, J. B. Wilkinson [9] similarly characterized the dimension function d_3.

THEOREM. *Let (X, ρ) be a metric space. Then $d_3(X, \rho) \leqq n$ if and only if every finite Lebesgue cover of X has an open refinement of order $\leqq n+1$.*

In this paper we continue the study of Lebesgue characterization of metric-dependent dimension functions. In §2 we give a Lebesgue cover characterization of d_2. In §3 and §4 we introduce two new metric-dependent dimension functions, d_6 and d_7, and characterize them in terms of Lebesgue covers.

DEFINITION. Let X be a set and $\mathcal{G} = \{\mathcal{G}_\lambda : \lambda \in \Lambda\}$ be a collection of collections of subsets of X. For each $\lambda \in \Lambda$, let $\mathcal{G}_\lambda = \{G_\alpha : \alpha \in A_\lambda\}$. Then

$$\bigwedge_{\lambda \in \Lambda} \{\mathcal{G}_\lambda\} = \{\bigcap G_{\alpha(\lambda)} : \alpha(\lambda) \in A_\lambda, \lambda \in \Lambda\}.$$

DEFINITION. Throughout this paper J will denote the set $\{1, 2, \cdots, n+1\}$ and $J' = J \cup \{n+2\}$, where the integer n will always be understood.

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2. **Characterization of d_2.** The reader is referred to the papers by Nagami and Roberts [7] and by Hodel [2] for the definitions of the dimension functions d_0, d_2, d_3, and d_5. Note that in some papers d_0 and μ dim are synonomous.

DEFINITION 2.1. Let $\mathcal{G}=\{G_\alpha: \alpha\in A\}$ be a cover of a metric space (X, ρ). We say that $\mathcal{G}$ is *uniformly shrinkable* if there exists a real number $\delta>0$ and a cover $\mathcal{F}=\{F_\alpha: \alpha\in A\}$ such that

(1) $F_\alpha\subset G_\alpha$ for all $\alpha\in A$.

(2) $\rho(F_\alpha, X-G_\alpha)>\delta$ for all $\alpha\in A$.

THEOREM 2.2. *Let $\mathcal{G}$ be a cover of a metric space (X, ρ). Then $\mathcal{G}$ is a Lebesgue cover of X if and only if $\mathcal{G}$ is uniformly shrinkable.*

PROOF (NECESSITY). Let $\mathcal{G}=\{G_\alpha: \alpha\in A\}$ be a Lebesgue cover of (X, ρ) with Lebesgue number $\delta>0$. Define for each $\alpha\in A$,

$$F_\alpha = \{x\in X: \rho(x, (X-G_\alpha)) \geqq \delta/3\}.$$

Clearly $F_\alpha\subset G_\alpha$ for all $\alpha\in A$. Let $x\in X$. Since $\mathcal{G}$ is Lebesgue, $S(x, \delta/2)\subset G_\beta$ for some $\beta\in A$. Hence $x\in F_\beta$ so that $\mathcal{F}=\{F_\alpha: \alpha\in A\}$ covers X. Note that $\mathcal{F}$ is actually a Lebesgue cover, since $S(x, \delta/6)\subset F_\beta$ above.

(SUFFICIENCY). Suppose $\mathcal{G}=\{G_\alpha: \alpha\in A\}$ is uniformly shrinkable to $\mathcal{F}=\{F_\alpha: \alpha\in A\}$, where $\rho(F_\alpha, X-G_\alpha)>\delta$ for all $\alpha\in A$. Let $x\in X$. Since $\mathcal{F}$ covers X, $x\in F_\beta$ for some $\beta\in A$. Therefore, $S(x, \delta)\subset G_\beta$, and hence $\mathcal{G}$ is Lebesgue.

CONSTRUCTION LEMMA. *Let X be a normal space, $\{G_\alpha: \alpha\in A\}$ a locally finite open collection, and $\{F_\alpha: \alpha\in A\}$ a closed collection such that $F_\alpha\subset G_\alpha$ for all $\alpha\in A$. If $\mathcal{G}=\bigwedge_{\alpha\in A}\{G_\alpha, X-F_\alpha\}$ has an open refinement of order $\leqq n+1$, then there exist closed sets B_α separating F_α and $X-G_\alpha$ for each $\alpha\in A$ such that* ord$\{B_\alpha: \alpha\in A\}\leqq n$.

PROOF. The proof proceeds essentially the same as the proof of [8, II, 5, B].

THEOREM 2.3. *Let (X, ρ) be a metric space. Then $d_2(X, \rho)\leqq n$ if and only if for every collection $\{\mathcal{G}_i: i\in J\}$ of $n+1$ binary Lebesgue covers of X, the cover $\mathcal{G}=\bigwedge_{i\in J}\mathcal{G}_i$ of X has an open refinement of order $\leqq n+1$.*

PROOF (NECESSITY). Suppose $d_2(X, \rho)\leqq n$ and let $\mathcal{G}_i=\{G_i, X-F_i\}$, $i\in J$, be a collection of $n+1$ binary Lebesgue covers of X. Let $\mathcal{G}=\bigwedge_{i\in J}\mathcal{G}_i$. We may assume each $\mathcal{G}_i$ to be an open cover by Theorem 2.2. It is clear that $\rho(F_i, X-G_i)>0$ for $i\in J$. Since $d_2(X, \rho)\leqq n$, there exist for each $i\in J$, open subsets U_i of X such that

(1) $F_i\subset U_i\subset(U_i)^-\subset G_i$.

(2) ord$\{((U_i)^- - U_i): i\in J\}\leqq n$.

Define $\mathcal{W}_0 = \bigwedge_{i \in J} \{U_i, X-(U_i)^-\}$. Clearly $\mathcal{W}_0$ satisfies

(1) $\mathcal{W}_0$ covers $X - \bigcup_{i \in J} B_i$ where $B_i = (U_i)^- - U_i$.

(2) $W \in \mathcal{W}_0$ implies there exists $G \in \mathcal{G}$ such that $W \subseteq G$.

(3) $\operatorname{ord}(\mathcal{W}_0) \leqq 1$.

(4) $W_1, W_2 \in \mathcal{W}_0$ implies $(W_1)^- \cap W_2 = \varnothing = W_1 \cap (W_2)^-$ if $W_1 \neq W_2$.

Step 1. Define $J_i = J - \{i\}$, $J_{i,j} = J - \{i, j\}$, etc. Also let

$$\mathcal{E}_i = \bigwedge_{j \in J_i} \{U_j, X-(U_j)^-\} \quad \text{and} \quad \mathcal{V}_1 = \{B_i \cap E \colon E \in \mathcal{E}_i,\ i \in J\}.$$

Note that $\mathcal{V}_1$ is a partition of all points of order 1 with respect to $\{B_i\colon i \in J\}$. Also for different V and V' in $\mathcal{V}_1$ we have as in (4) above $\overline{V} \cap V' = \varnothing = V \cap (V')^-$.

Since X is completely normal and $\mathcal{V}_1$ is finite, there exists a collection $\mathcal{W}_1$ of pairwise disjoint open subsets of X each containing one member of $\mathcal{V}_1$. Hence $\operatorname{ord}(\mathcal{W}_1) \leqq 1$, so that $\operatorname{ord}(\mathcal{W}_0 \cup \mathcal{W}_1) \leqq 2$. Also we may assume that $W \in \mathcal{W}_1$ implies there exists a $G \in \mathcal{G}$ such that $W \subseteq G$. Otherwise, we intersect W with G to obtain this property.

Step 2. As in Step 1 define $\mathcal{E}_{i,j} = \bigwedge_{k \in J_{i,j}} \{U_k, X-(U_k)^-\}$ and $\mathcal{V}_2 = \{(B_i \cap B_j) \cap E\colon E \in \mathcal{E}_{i,j},\ i, j \in J,\ i \neq j\}$. As before $\mathcal{V}_2$ is a partition of all points of order 2 with respect to $\{B_i\colon i \in J\}$ such that for different V and V' in $\mathcal{V}_2$, we have again $\overline{V} \cap V' = \varnothing = V \cap (V')^-$. Thus there exists a collection $\mathcal{W}_2$ of pairwise disjoint open subsets of X each containing one member of $\mathcal{V}_2$. Therefore, $\operatorname{ord}(\mathcal{W}_2) \leqq 1$, and hence $\operatorname{ord}(\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2) \leqq 3$. We may assume $W \subseteq G$ for every $W \in \mathcal{W}_2$ and for some $G \in \mathcal{G}$.

Now continue this process through step n, and define $\mathcal{W} = \bigcup_{i=0}^{n} \mathcal{W}_i$. Since $\operatorname{ord}\{B_i\colon i \in J\} \leqq n$, $\mathcal{W}$ covers X. Also $\mathcal{W} < \mathcal{G}$ and $\operatorname{ord}(\mathcal{W}) \leqq n+1$ by construction. Therefore $\mathcal{W}$ is the desired open cover.

(SUFFICIENCY). Let $\{C_i, C_i'\colon i \in J\}$ be a collection of $n+1$ pairs of disjoint closed sets such that $\rho(C_i, C_i') > 0$ for $i \in J$. Since $\mathcal{G}_i = \{X - C_i, X - C_i'\}$ is a binary Lebesgue cover of X, $\mathcal{G} = \bigwedge_{i \in J} \mathcal{G}_i$ has a refinement of order $\leqq n+1$. By the Construction Lemma there exist closed sets B_i separating C_i and C_i' such that $\operatorname{ord}\{B_i\colon i \in J\} \leqq n$. Hence $d_2(X, \rho) \leqq n$.

THEOREM 2.4. *Let (X, ρ) be a metric space. Then $d_2(X, \rho) \leqq n$ if and only if every Lebesgue cover $\mathcal{G} = \{G_1, G_2, \cdots, G_{n+2}\}$ of X consisting of $n+2$ members has an open refinement of order $\leqq n+1$.*

PROOF (NECESSITY). Suppose $d_2(X, \rho) \leqq n$, and let $\mathcal{G} = \{G_1, G_2, \cdots, G_{n+2}\}$ be a Lebesgue cover of X with Lebesgue number $\delta > 0$. As in Theorem 2.2 define

$$F_i = \{x \in X\colon \rho(x, X - G_i) \geqq \delta/3\}$$

for each $i \in J'$, so that $\mathcal{F} = \{F_1, F_2, \cdots, F_{n+2}\}$ is a uniform shrink of $\mathcal{G}$. Since $\{G_i, X-F_i\}$ is a Lebesgue cover of X for $i \in J$, by Theorem 2.3 $\mathcal{G}^* = \bigwedge_{i \in J}\{G_i, X-F_i\}$ has an open refinement $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{U}) \leqq n+1$. But $\mathcal{G}^*$ refines $\mathcal{G}$ since $\mathcal{F}$ covers X. Hence $\mathcal{U}$ is the desired open cover.

(SUFFICIENCY). Let $\{C_i, C_i' : i \in J\}$ be a collection of $n+1$ pairs of disjoint closed sets such that $\rho(C_i, C_i') = \delta_i > 0$ for $i \in J$. Define $\delta = \min(\delta_i: i \in J)$, $G_i = S(C_i, \delta/2)$, $H_i = (S(C_i, \delta/4))^-$ for $i \in J$, and $G_{n+2} = X - \bigcup_{i \in J} H_i$. Since $\mathcal{G} = \{G_1, G_2, \cdots, G_{n+2}\}$ is a Lebesgue cover of X by construction, $\mathcal{G}$ has an open refinement $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $\operatorname{ord}(\mathcal{U}) \leqq n+1$. Let f be the function, $f: A \to J'$, defined by $f(\alpha) =$ smallest integer $i \in J'$ such that $U_\alpha \subseteq G_i$. Now define $U_i = \bigcup\{U_\alpha : f(\alpha) = i\}$ for $i \in J'$. Hence we may assume that $\mathcal{U} = \{U_1, U_2, \cdots, U_{n+2}\}$ with the order unchanged. Define

$$E_i = \{x \in C_i : x \notin U_i\}, \qquad S_i = S(E_i, \delta/8), \qquad V_i = U_i \cup S_i$$

for $i \in J$, and $V_{n+2} = U_{n+2}$. Since $S_i \cap G_{n+2} = \varnothing$ for $i \in J$, then $\mathcal{V} = \{V_1, V_2, \cdots, V_{n+2}\}$ is an open cover of X such that $\operatorname{ord}(\mathcal{V}) \leqq n+1$ and $C_i \subset V_i$ for $i \in J$. Since $\mathcal{V}$ is finite there exists a closed cover $\mathcal{F} = \{F_1, F_2, \cdots, F_{n+2}\}$ of X such that $C_i \subseteq F_i \subseteq V_i$ for $i \in J'$ [**5**, Lemma 1.5]. Thus X normal implies there exist open sets W_i such that $F_i \subset W_i \subset (W_i)^- \subset V_i$ for $i \in J$. Define $B_i = (W_i)^- - W_i$ for $i \in J$. Clearly B_i separates C_i from C_i' for $i \in J$. We assert $\bigcap_{i \in J} B_i = \varnothing$. Suppose there exists a point $x \in \bigcap_{i \in J} B_i$. Then $x \notin F_i$ for $i \in J$. Hence $x \in F_{n+2} \subset V_{n+2}$. But $x \in V_i$ for $i \in J$, so that $x \in \bigcap_{i=1}^{n+2} V_i$. This is a contradiction since $\operatorname{ord}(\mathcal{V}) \leqq n+1$. Hence $d_2(X, \rho) \leqq n$.

3. The dimension function d_6.

DEFINITION 3.1. Let (X, ρ) be a metric space. If $X = \varnothing$, $d_6(X, \rho) = -1$. Otherwise, $d_6(X, \rho) \leqq n$ if (X, ρ) satisfies this condition:

(D_6) Given any countable collection of closed pairs $\{C_i, C_i' : i = 1, 2, \cdots\}$ such that there exists $\delta > 0$ with

(1) $\rho(C_i, C_i') > \delta$ for all i,

(2) $\{X - C_i' : i = 1, 2, \cdots\}$ is locally finite,

then there exist closed sets B_i separating C_i from C_i' such that $\operatorname{ord}\{B_i : i = 1, 2, \cdots\} \leqq n$. If $d_6(X, \rho) \leqq n$ is true and $d_6(X, \rho) \leqq n-1$ is false, then $d_6(X, \rho) = n$.

Note that $d_6(X, \rho) \leqq d_5(X, \rho)$ by definition.

THEOREM 3.2. *Let (X, ρ) be a metric space. Then $d_6(X, \rho) \leqq n$ if and only if every countable, locally finite Lebesgue cover has an open refinement of order $\leqq n+1$.*

PROOF (NECESSITY). This proof is exactly the same as the proof of the necessity of Theorem 4.2 below and hence is omitted.

(SUFFICIENCY). Let $\{C_i, C_i' : i=1, 2, \cdots\}$ be a countable collection of closed pairs satisfying property (D_6). Since $\{X-C_i' : i=1, 2, \cdots\}$ is locally finite, $\mathcal{G}=\bigwedge_{i=1}^{\infty}\{X-C_i, X-C_i'\}$ is a countable locally finite Lebesgue cover of X. Thus $\mathcal{G}$ has an open refinement of order $\leqq n+1$ and hence by the Construction Lemma, $d_6(X, \rho)\leqq n$.

THEOREM 3.3. *Let (X, ρ) be a metric space. Every countable Lebesgue cover of X has a countable locally finite Lebesgue refinement.*

PROOF. This proof is a modification of [2, Lemma 2.2]. Let $\mathcal{G}=\{G_1, G_2, \cdots\}$ be a Lebesgue cover of X with Lebesgue number $\delta>0$. Define $F_i=\{x\in X: \rho(x, X-G_i)\geqq\delta/2\}$ for all i. Then $\mathcal{F}=\{F_1, F_2, \cdots\}$ covers X as before. Define $U_i=G_i-\bigcup_{j<i}(S(F_j, \delta/8))^-$ for all i, and $\mathcal{U}=\{U_1, U_2, \cdots\}$. Clearly $\mathcal{U}$ refines $\mathcal{G}$ in a 1-1 manner. We assert that $\mathcal{U}$ is a locally finite Lebesgue cover of X.

(1) Let $x\in X$. Choose the smallest i such that $x\in(S(F_i, \delta/8))^-$. Then $x\in G_i-\bigcup_{j<i}(S(F_j, \delta/8))^-=U_i$. Hence $\mathcal{U}$ covers x.

(2) Let $x\in X$. Choose the smallest i such that $x\in S(F_i, \delta/8)$. Then $S(F_i, \delta/8)\cap U_j=\emptyset$ for $j>i$, so that $\mathcal{U}$ is locally finite.

(3) Let $x\in X$. Choose the smallest i such that $S(x, \delta/8)\cap(S(F_i, \delta/8))^-\neq\emptyset$. Then $S(x, \delta/8)\subset G_i-\bigcup_{j<i}(S(F_j, \delta/8))^-=U_i$. Hence $\mathcal{U}$ is Lebesgue.

From Theorem 3.2 and Theorem 3.3 we have the following.

THEOREM 3.4. *Let (X, ρ) be a metric space. Then $d_6(X, \rho)\leqq n$ if and only if every countable Lebesgue cover has an open refinement of order $\leqq n+1$.*

COROLLARY 3.5. *Let (X, ρ) be a separable metric space. Then $d_6(X, \rho)=d_0(X, \rho)$.*

COROLLARY 3.6 (HODEL). *Let (X, ρ) be a separable metric space. Then $d_5(X, \rho)=d_0(X, \rho)$.*

4. The dimension function d_7.

DEFINITION 4.1. Let (X, ρ) be a metric space. If $X=\emptyset$, then $d_7(X, \rho)=-1$. Otherwise, $d_7(X, \rho)\leqq n$ if (X, ρ) satisfies this condition:

(D_7) Given any collection of closed pairs $\{C_\alpha, C_\alpha' : \alpha\in A\}$ such that there exists $\delta>0$ with

(1) $\rho(C_\alpha, C_\alpha')>\delta$ for all $\alpha\in A$,

(2) $\{X-C_\alpha' : \alpha\in A\}$ is locally finite,

then there exist closed sets B_α separating C_α and C_α' such that $\operatorname{ord}\{B_\alpha: \alpha\in A\}\leqq n$. If $d_7(X, \rho)\leqq n$ is true, and $d_7(X, \rho)\leqq n-1$ is false, then $d_7(X, \rho)=n$.

THEOREM 4.2. *Let (X, ρ) be a metric space. Then $d_7(X, \rho)\leqq n$ if and only if every locally finite Lebesgue cover has a refinement of order $\leqq n+1$.*

PROOF (NECESSITY). Suppose $d_7(X, \rho) \leqq n$, and let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a locally finite Lebesgue cover of X with Lebesgue number $\delta > 0$. By Theorem 2.2 $\mathcal{G}$ is uniformly shrinkable to a closed cover $\mathfrak{F} = \{F_\alpha : \alpha \in A\}$ such that $F_\alpha \subset G_\alpha$ and $\rho(F_\alpha, X - G_\alpha) \geqq \delta/3$ for all $\alpha \in A$. Since $d_7(X, \rho) \leqq n$ there exist closed sets B_α and open sets U_α and U'_α which satisfy the following conditions:

(1) B_α separates F_α and $X - G_\alpha$ for all $\alpha \in A$.

(2) $X - B_\alpha = U_\alpha \cup U'_\alpha$ for all $\alpha \in A$.

(3) $U_\alpha \cap U'_\alpha = \emptyset$ for all $\alpha \in A$.

(4) $\operatorname{ord}\{B_\alpha : \alpha \in A\} \leqq n$.

Since X is paracompact there exist open sets V_α such that $B_\alpha \subset V_\alpha \subset G_\alpha$ for all $\alpha \in A$, and $\operatorname{ord}\{V_\alpha : \alpha \in A\} \leqq n$ [6, Theorem 1.3]. Let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$. Define $\mathcal{U} = \bigwedge_{\alpha \in A}\{U_\alpha, U'_\alpha\}$ which is a locally finite open cover of $X - \bigcup_{\alpha \in A} B_\alpha$, since $\mathcal{G}$ is locally finite. Also $\operatorname{ord}(\mathcal{U}) \leqq 1$, and $U \in \mathcal{U}$ implies that there exist some $G_\beta \in \mathcal{G}$ such that $U \subset G_\beta$. Thus $\mathcal{G}^* = \mathcal{U} \cup \mathcal{V}$ is an open refinement of $\mathcal{G}$ and $\operatorname{ord}(\mathcal{G}^*) \leqq n+1$.

(SUFFICIENCY). Suppose every locally finite Lebesgue cover of X has an open refinement of order $\leqq n+1$. Let $\{C_\alpha, C'_\alpha : \alpha \in A\}$ be any collection of closed pairs satisfying condition (D_7) above. Then $\{X - C_\alpha, X - C'_\alpha\}$ is a binary open Lebesgue cover of X for each $\alpha \in A$. Since $\{X - C'_\alpha : \alpha \in A\}$ is locally finite, $\mathcal{G} = \bigwedge_{\alpha \in A}\{X - C_\alpha, X - C'_\alpha\}$ is a locally finite Lebesgue cover of X. Hence $\mathcal{G}$ has an open refinement of order $\leqq n+1$. Therefore, by the Construction Lemma there exist closed sets B_α such that B_α separates C_α and C'_α for each $\alpha \in A$ and $\operatorname{ord}\{B_\alpha : \alpha \in A\} \leqq n$. Hence $d_7(X, \rho) \leqq n$.

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