1. **Introduction.** Let $(X, \rho)$ be a metric space, let $\dim(X)$ be the covering dimension of $X$, and let $d_0(X, \rho)$ be the metric dimension of $X$. Let $d_2$ and $d_3$ denote the metric-dependent dimension functions for metric spaces introduced by Nagami and Roberts [7], and let $d_5$ be the metric-dependent dimension function defined by Hodel [2]. A summary of the relationships among these dimension functions for $(X, \rho)$ is the following:

$$d_2(X, \rho) \leq d_3(X, \rho) \leq d_5(X, \rho) \leq d_0(X, \rho) \leq \dim(X) \leq 2d_5(X, \rho).$$

In 1957, V. I. Egorov [1] characterized the dimension function $d_0$ in terms of Lebesgue covers as follows:

**Theorem.** Let $(X, \rho)$ be a metric space. Then $d_0(X, \rho) \leq n$ if and only if every Lebesgue cover of $X$ has an open refinement of order $\leq n + 1$.


**Theorem.** Let $(X, \rho)$ be a metric space. Then $d_5(X, \rho) \leq n$ if and only if every finite Lebesgue cover of $X$ has an open refinement of order $\leq n + 1$.

In this paper we continue the study of Lebesgue characterization of metric-dependent dimension functions. In §2 we give a Lebesgue cover characterization of $d_2$. In §3 and §4 we introduce two new metric-dependent dimension functions, $d_6$ and $d_7$, and characterize them in terms of Lebesgue covers.

**Definition.** Let $X$ be a set and $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ be a collection of collections of subsets of $X$. For each $\lambda \in \Lambda$, let $G_\lambda = \{G_\alpha : \alpha \in A_\lambda\}$. Then

$$\bigwedge_{\lambda \in \Lambda} \{G_\lambda\} = \{\bigcap G_{\alpha(\lambda)} : \alpha(\lambda) \in A_\lambda, \lambda \in \Lambda\}.$$

**Definition.** Throughout this paper $J$ will denote the set $\{1, 2, \cdots, n+1\}$ and $J' = J \cup \{n+2\}$, where the integer $n$ will always be understood.

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2. Characterization of $d_2$. The reader is referred to the papers by Nagami and Roberts [7] and by Hodel [2] for the definitions of the dimension functions $d_0$, $d_2$, $d_3$, and $d_5$. Note that in some papers $d_0$ and $\mu \dim$ are synonymous.

**Definition 2.1.** Let $\mathcal{G} = \{ G_\alpha : \alpha \in A \}$ be a cover of a metric space $(X, \rho)$. We say that $\mathcal{G}$ is uniformly shrinkable if there exists a real number $\delta > 0$ and a cover $\mathcal{F} = \{ F_\alpha : \alpha \in A \}$ such that

1. $F_\alpha \subset G_\alpha$ for all $\alpha \in A$.
2. $\rho(F_\alpha, X - G_\alpha) > \delta$ for all $\alpha \in A$.

**Theorem 2.2.** Let $\mathcal{G}$ be a cover of a metric space $(X, \rho)$. Then $\mathcal{G}$ is a Lebesgue cover of $X$ if and only if $\mathcal{G}$ is uniformly shrinkable.

**Proof (Necessity).** Let $\mathcal{G} = \{ G_\alpha : \alpha \in A \}$ be a Lebesgue cover of $(X, \rho)$ with Lebesgue number $\delta > 0$. Define for each $\alpha \in A$,

$$F_\alpha = \{ x \in X : \rho(x, (X - G_\alpha)) \geq \delta/3 \}.$$  

Clearly $F_\alpha \subset G_\alpha$ for all $\alpha \in A$. Let $x \in X$. Since $\mathcal{G}$ is Lebesgue, $S(x, \delta/2) \subset G_\beta$ for some $\beta \in A$. Hence $x \in F_\beta$ so that $\mathcal{F} = \{ F_\alpha : \alpha \in A \}$ covers $X$. Note that $\mathcal{F}$ is actually a Lebesgue cover, since $S(x, \delta/6) \subset F_\beta$ above.

**Sufficiency.** Suppose $\mathcal{G} = \{ G_\alpha : \alpha \in A \}$ is uniformly shrinkable to $\mathcal{F} = \{ F_\alpha : \alpha \in A \}$, where $\rho(F_\alpha, X - G_\alpha) > \delta$ for all $\alpha \in A$. Let $x \in X$. Since $\mathcal{F}$ covers $X$, $x \in F_\beta$ for some $\beta \in A$. Therefore, $S(x, \delta) \subset G_\beta$, and hence $\mathcal{G}$ is Lebesgue.

**Construction Lemma.** Let $X$ be a normal space, $\{ G_\alpha : \alpha \in A \}$ a locally finite open collection, and $\{ F_\alpha : \alpha \in A \}$ a closed collection such that $F_\alpha \subset G_\alpha$ for all $\alpha \in A$. If $\mathcal{G} = \bigwedge_{\alpha \in A} \{ G_\alpha, X - F_\alpha \}$ has an open refinement of order $\leq n + 1$, then there exist closed sets $B_\alpha$ separating $F_\alpha$ and $X - G_\alpha$ for each $\alpha \in A$ such that $\operatorname{ord} \{ B_\alpha : \alpha \in A \} \leq n$.

**Proof.** The proof proceeds essentially the same as the proof of [8, II, 5, B].

**Theorem 2.3.** Let $(X, \rho)$ be a metric space. Then $d_2(X, \rho) \leq n$ if and only if for every collection $\{ G_i : i \in J \}$ of $n + 1$ binary Lebesgue covers of $X$, the cover $\mathcal{G} = \bigwedge_{i \in J} G_i$ of $X$ has an open refinement of order $\leq n + 1$.

**Proof (Necessity).** Suppose $d_2(X, \rho) \leq n$ and let $\mathcal{G}_i = \{ G_i, X - F_i \}$, $i \in J$, be a collection of $n + 1$ binary Lebesgue covers of $X$. Let $\mathcal{G} = \bigwedge_{i \in J} \mathcal{G}_i$. We may assume each $\mathcal{G}_i$ to be an open cover by Theorem 2.2. It is clear that $\rho(F_i, X - G_i) > 0$ for $i \in J$. Since $d_2(X, \rho) \leq n$, there exist for each $i \in J$, open subsets $U_i$ of $X$ such that

1. $F_i \subset U_i \subset (U_i)^- \subset G_i$.
2. $\operatorname{ord} \{ ((U_i)^- \setminus U_i) : i \in J \} \leq n$. 
Define \( \mathcal{W}_0 = \bigwedge_{i \in J} \{ U_i, X - (U_i)^- \} \). Clearly \( \mathcal{W}_0 \) satisfies

1. \( \mathcal{W}_0 \) covers \( X - \bigcup_{i \in J} B_i \) where \( B_i = (U_i)^- - U_i \).
2. \( W \in \mathcal{W}_0 \) implies there exists \( G \in \mathcal{G} \) such that \( W \subseteq G \).
3. \( \text{ord}(\mathcal{W}_0) \leq 1 \).
4. \( W_1, W_2 \in \mathcal{W}_0 \) implies \( (W_1)^- \cap W_2 = \emptyset = W_1 \cap (W_2)^- \) if \( W_1 \neq W_2 \).

**Step 1.** Define \( J_i = J - \{ i \}, J_{i,j} = J - \{ i, j \} \), etc. Also let

\[
\mathcal{E}_i = \bigwedge_{j \in J_i} \{ U_j, X - (U_j)^- \} \quad \text{and} \quad \mathcal{U}_1 = \{ B_i \cap E : E \in \mathcal{E}_i, i \in J \}.
\]

Note that \( \mathcal{U}_1 \) is a partition of all points of order 1 with respect to \( \{ B_i : i \in J \} \). Also for different \( V \) and \( V' \) in \( \mathcal{U}_1 \) we have as in (4) above \( V \cap V' = \emptyset = V \cap (V')^- \).

Since \( X \) is completely normal and \( \mathcal{U}_1 \) is finite, there exists a collection \( \mathcal{W}_1 \) of pairwise disjoint open subsets of \( X \) each containing one member of \( \mathcal{U}_1 \). Hence \( \text{ord}(\mathcal{W}_1) \leq 1 \), so that \( \text{ord}(\mathcal{W}_0 \cup \mathcal{W}_1) \leq 2 \). Also we may assume that \( W \in \mathcal{W}_1 \) implies there exists a \( G \in \mathcal{G} \) such that \( W \subseteq G \). Otherwise, we intersect \( W \) with \( G \) to obtain this property.

**Step 2.** As in Step 1 define \( \mathcal{E}_{i,j} = \bigwedge_{k \in J_{i,j}} \{ U_k, X - (U_k)^- \} \) and \( \mathcal{U}_2 = \{ (B_i \cap B_j) \cap E : E \in \mathcal{E}_{i,j}, i, j \in J, i \neq j \} \). As before \( \mathcal{U}_2 \) is a partition of all points of order 2 with respect to \( \{ B_i : i \in J \} \) such that for different \( V \) and \( V' \) in \( \mathcal{U}_2 \), we have again \( V \cap V' = \emptyset = V \cap (V')^- \). Thus there exists a collection \( \mathcal{W}_2 \) of pairwise disjoint open subsets of \( X \) each containing one member of \( \mathcal{U}_2 \). Therefore, \( \text{ord}(\mathcal{W}_2) \leq 1 \), and hence \( \text{ord}(\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2) \leq 3 \). We may assume \( W \subseteq G \) for every \( W \in \mathcal{W}_2 \) and for some \( G \in \mathcal{G} \).

Now continue this process through step \( n \), and define \( \mathcal{W} = \bigcup_{i=0}^n \mathcal{W}_i \). Since \( \text{ord} \{ B_i : i \in J \} \leq n \), \( \mathcal{W} \) covers \( X \). Also \( \mathcal{W} \subseteq \mathcal{G} \) and \( \text{ord}(\mathcal{W}) \leq n + 1 \) by construction. Therefore \( \mathcal{W} \) is the desired open cover.

**Sufficiency.** Let \( \{ C_i, C'_i : i \in J \} \) be a collection of \( n+1 \) pairs of disjoint closed sets such that \( \rho(C_i, C'_i) > 0 \) for \( i \in J \). Since \( \mathcal{G}_i = \{ X - C_i, X - C'_i \} \) is a binary Lebesgue cover of \( X \), \( \mathcal{G}_i = \bigwedge_{i \in J} \mathcal{G}_i \) has a refinement of order \( \leq n + 1 \). By the Construction Lemma there exist closed sets \( B_i \) separating \( C_i \) and \( C'_i \) such that \( \text{ord} \{ B_i : i \in J \} \leq n \). Hence \( d_2(X, \rho) \leq n \).

**Theorem 2.4.** Let \((X, \rho)\) be a metric space. Then \( d_2(X, \rho) \leq n \) if and only if every Lebesgue cover \( \mathcal{G} = \{ G_1, G_2, \ldots, G_{n+2} \} \) of \( X \) consisting of \( n+2 \) members has an open refinement of order \( \leq n + 1 \).

**Proof (Necessity).** Suppose \( d_2(X, \rho) \leq n \), and let \( \mathcal{G} = \{ G_1, G_2, \ldots, G_{n+2} \} \) be a Lebesgue cover of \( X \) with Lebesgue number \( \delta > 0 \). As in Theorem 2.2 define

\[
F_i = \{ x \in X : \rho(x, X - G_i) \geq \delta/3 \}
\]
for each $i \in J'$, so that $\mathcal{F} = \{ F_1, F_2, \ldots, F_{n+2} \}$ is a uniform shrink of $\mathcal{G}$. Since $\{ G_i, X - F_i \}$ is a Lebesgue cover of $X$ for $i \in J$, by Theorem 2.3 $\mathcal{G}^* = \bigwedge_{i \in J} \{ G_i, X - F_i \}$ has an open refinement $\mathcal{U}$ such that $\text{ord}(\mathcal{U}) \leq n + 1$. But $\mathcal{G}^*$ refines $\mathcal{G}$ since $\mathcal{F}$ covers $X$. Hence $\mathcal{U}$ is the desired open cover.

**Sufficiency.** Let $\{ C_i, C_i' : i \in J \}$ be a collection of $n + 1$ pairs of disjoint closed sets such that $\rho(C_i, C_i') = \delta_i > 0$ for $i \in J$. Define $\delta = \min(\delta_i : i \in J)$, $G_i = S(C_i, \delta/2)$, $H_i = (S(C_i, \delta/4))$. Since $\mathcal{G} = \{ G_1, G_2, \ldots, G_{n+2} \}$ is a Lebesgue cover of $X$ by construction, $\mathcal{G}$ has an open refinement $\mathcal{U} = \{ U_\alpha : \alpha \in A \}$ such that $\text{ord}(\mathcal{U}) \leq n + 1$. Let $f$ be the function, $f : A \rightarrow J'$, defined by $f(\alpha) =$ smallest integer $i \in J'$ such that $U_\alpha \subset G_i$. Now define $U_i = \bigcup \{ U_\alpha : f(\alpha) = i \}$ for $i \in J'$. Hence we may assume that $\mathcal{U} = \{ U_1, U_2, \ldots, U_{n+2} \}$ with the order unchanged. Define

$$E_i = \{ x \in C_i : x \notin U_i \}, \quad S_i = S(E_i, \delta/8), \quad V_i = U_i \cup S_i$$

for $i \in J$, and $V_{n+2} = U_{n+2}$. Since $S_i \cap G_{n+2} = \emptyset$ for $i \in J$, then $\mathcal{V} = \{ V_1, V_2, \ldots, V_{n+2} \}$ is an open cover of $X$ such that $\text{ord}(\mathcal{V}) \leq n + 1$ and $C_i \subset V_i$ for $i \in J$. Since $\mathcal{V}$ is finite there exists a closed cover $\mathcal{F} = \{ F_1, F_2, \ldots, F_{n+2} \}$ of $X$ such that $C_i \subset F_i \subset V_i$ for $i \in J'$ [5, Lemma 1.5]. Thus $X$ normal implies there exist open sets $W_i$ such that $F_i \subset W_i \subset W_i^c \subset V_i$ for $i \in J$. Define $B_i = (W_i)^c - W_i$ for $i \in J$. Clearly $B_i$ separates $C_i$ from $C_i'$ for $i \in J$. We assert $\bigcap_{i \in J} B_i = \emptyset$. Suppose there exists a point $x \in \bigcap_{i \in J} B_i$. Then $x \in F_i$ for $i \in J$. Hence $x \in V_i$ for $i \in J$, so that $x \in \bigcap_{i=1}^{n+2} V_i$. This is a contradiction since $\text{ord}(\mathcal{V}) \leq n + 1$. Hence $d_6(X, \rho) \leq n$.

3. The dimension function $d_6$.

**Definition 3.1.** Let $(X, \rho)$ be a metric space. If $X = \emptyset$, $d_6(X, \rho) = -1$. Otherwise, $d_6(X, \rho) \leq n$ if $(X, \rho)$ satisfies this condition:

(1) $\rho(C_i, C_i') > \delta$ for all $i$,

(2) $\{ X - C_i' : i = 1, 2, \ldots \}$ is locally finite,

then there exist closed sets $B_i$ separating $C_i$ from $C_i'$ such that $\text{ord}\{ B_i : i = 1, 2, \ldots \} \leq n$. If $d_6(X, \rho) \leq n$ is true and $d_6(X, \rho) \leq n - 1$ is false, then $d_6(X, \rho) = n$.

Note that $d_6(X, \rho) \leq d_5(X, \rho)$ by definition.

**Theorem 3.2.** Let $(X, \rho)$ be a metric space. Then $d_6(X, \rho) \leq n$ if and only if every countable, locally finite Lebesgue cover has an open refinement of order $\leq n + 1$.

**Proof (Necessity).** This proof is exactly the same as the proof of the necessity of Theorem 4.2 below and hence is omitted.
(Sufficiency). Let \( \{ C_i, C'_i : i = 1, 2, \ldots \} \) be a countable collection of closed pairs satisfying property \((D_6)\). Since \( \{ X - C'_i : i = 1, 2, \ldots \} \) is locally finite, \( \mathcal{G} = \mathbb{N} \cup \{ X - C_i, X - C'_i \} \) is a countable locally finite Lebesgue cover of \( X \). Thus \( \mathcal{G} \) has an open refinement of order \( \leq n + 1 \) and hence by the Construction Lemma, \( d_6(X, \rho) \leq n \).

**Theorem 3.3.** Let \( (X, \rho) \) be a metric space. Every countable Lebesgue cover of \( X \) has a countable locally finite Lebesgue refinement.

**Proof.** This proof is a modification of [2, Lemma 2.2]. Let \( \mathcal{G} = \{ G_1, G_2, \ldots \} \) be a Lebesgue cover of \( X \) with Lebesgue number \( \delta > 0 \). Define \( F_i = \{ x \in X : \rho(x, X - G_i) \geq \delta/2 \} \) for all \( i \). Then \( \mathcal{F} = \{ F_1, F_2, \ldots \} \) covers \( X \) as before. Define \( U_i = G_i - \cup_{j<i}(S(F_j, \delta/8))^{-} \) for all \( i \), and \( \mathcal{U} = \{ U_1, U_2, \ldots \} \). Clearly \( \mathcal{U} \) refines \( \mathcal{G} \) in a 1-1 manner. We assert that \( \mathcal{U} \) is a locally finite Lebesgue cover of \( X \).

1. Let \( x \in X \). Choose the smallest \( i \) such that \( x \in (S(F_i, \delta/8))^{-} \). Then \( x \in G_i - \cup_{j<i}(S(F_j, \delta/8))^{-} = U_i \). Hence \( \mathcal{U} \) covers \( x \).
2. Let \( x \in X \). Choose the smallest \( i \) such that \( x \in S(F_i, \delta/8) \). Then \( S(F_i, \delta/8) \cap U_j = \emptyset \) for \( j > i \), so that \( \mathcal{U} \) is locally finite.
3. Let \( x \in X \). Choose the smallest \( i \) such that \( S(x, \delta/8) \cap (S(F_i, \delta/8))^{-} \neq \emptyset \). Then \( S(x, \delta/8) \subseteq G_i - \cup_{j<i}(S(F_j, \delta/8))^{-} = U_i \). Hence \( \mathcal{U} \) is Lebesgue.

From Theorem 3.2 and Theorem 3.3 we have the following.

**Theorem 3.4.** Let \( (X, \rho) \) be a metric space. Then \( d_6(X, \rho) \leq n \) if and only if every countable Lebesgue cover has an open refinement of order \( \leq n + 1 \).

**Corollary 3.5.** Let \( (X, \rho) \) be a separable metric space. Then \( d_6(X, \rho) = d_0(X, \rho) \).

**Corollary 3.6 (Hodel).** Let \( (X, \rho) \) be a separable metric space. Then \( d_6(X, \rho) = d_6(X, \rho) \).

4. The dimension function \( d_7 \).

**Definition 4.1.** Let \( (X, \rho) \) be a metric space. If \( X = \emptyset \), then \( d_7(X, \rho) = -1 \). Otherwise, \( d_7(X, \rho) \leq n \) if \( (X, \rho) \) satisfies this condition:

\((D_7)\) Given any collection of closed pairs \( \{ C_\alpha, C'_\alpha : \alpha \in A \} \) such that there exists \( \delta > 0 \) with

1. \( \rho(C_\alpha, C'_\alpha) > \delta \) for all \( \alpha \in A \),
2. \( \{ X - C'_\alpha : \alpha \in A \} \) is locally finite,
then there exist closed sets \( B_\alpha \) separating \( C_\alpha \) and \( C'_\alpha \) such that \( \text{ord} \{ B_\alpha : \alpha \in A \} \leq n \). If \( d_7(X, \rho) \leq n \) is true, and \( d_7(X, \rho) \leq n - 1 \) is false, then \( d_7(X, \rho) = n \).

**Theorem 4.2.** Let \( (X, \rho) \) be a metric space. Then \( d_7(X, \rho) \leq n \) if and only if every locally finite Lebesgue cover has a refinement of order \( \leq n + 1 \).
Proof (Necessity). Suppose \( d_\gamma(X, \rho) \leq n \), and let \( \mathcal{G} = \{ G_\alpha : \alpha \in A \} \) be a locally finite Lebesgue cover of \( X \) with Lebesgue number \( \delta > 0 \). By Theorem 2.2 \( \mathcal{G} \) is uniformly shrinkable to a closed cover \( \mathcal{F} = \{ F_\alpha : \alpha \in A \} \) such that \( F_\alpha \subseteq G_\alpha \) and \( \rho(F_\alpha, X - G_\alpha) \geq \delta/3 \) for all \( \alpha \in A \). Since \( d_\gamma(X, \rho) \leq n \) there exist closed sets \( B_\alpha \) and open sets \( U_\alpha \) and \( U_\alpha' \) which satisfy the following conditions:

1. \( B_\alpha \) separates \( F_\alpha \) and \( X - G_\alpha \) for all \( \alpha \in A \).
2. \( X - B_\alpha = U_\alpha \cup U_\alpha' \) for all \( \alpha \in A \).
3. \( U_\alpha \cap U_\alpha' = \emptyset \) for all \( \alpha \in A \).
4. \( \text{ord}\{ B_\alpha : \alpha \in A \} \leq n \).

Since \( X \) is paracompact there exist open sets \( V_\alpha \) such that \( B_\alpha \subseteq V_\alpha \subset G_\alpha \) for all \( \alpha \in A \), and \( \text{ord}\{ V_\alpha : \alpha \in A \} \leq n \) [6, Theorem 1.3]. Let \( \mathcal{U} = \{ V_\alpha : \alpha \in A \} \). Define \( \mathcal{U} = \bigwedge_{\alpha \in A} \{ U_\alpha, U_\alpha' \} \) which is a locally finite open cover of \( X - \bigcup_{\alpha \in A} B_\alpha \), since \( \mathcal{G} \) is locally finite. Also \( \text{ord}(\mathcal{U}) \leq 1 \), and \( U \subseteq \mathcal{U} \) implies that there exist some \( G_\beta \in \mathcal{G} \) such that \( U \subseteq G_\beta \). Thus \( \mathcal{G}^* = \mathcal{U} \cup \mathcal{U} \) is an open refinement of \( \mathcal{G} \) and \( \text{ord}(\mathcal{G}^*) \leq n + 1 \).

(Sufficiency). Suppose every locally finite Lebesgue cover of \( X \) has an open refinement of order \( \leq n + 1 \). Let \( \{ C_\alpha, C'_\alpha : \alpha \in A \} \) be any collection of closed pairs satisfying condition (D7) above. Then \( \{ X - C_\alpha, X - C'_\alpha \} \) is a binary open Lebesgue cover of \( X \) for each \( \alpha \in A \). Since \( \{ X - C'_\alpha : \alpha \in A \} \) is locally finite, \( \mathcal{G} = \bigwedge_{\alpha \in A} \{ X - C_\alpha, X - C'_\alpha \} \) is a locally finite Lebesgue cover of \( X \). Hence \( \mathcal{G} \) has an open refinement of order \( \leq n + 1 \). Therefore, by the Construction Lemma there exist closed sets \( B_\alpha \) such that \( B_\alpha \) separates \( C_\alpha \) and \( C'_\alpha \) for each \( \alpha \in A \) and \( \text{ord}\{ B_\alpha : \alpha \in A \} \leq n \). Hence \( d_\gamma(X, \rho) \leq n \).

References


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