HOMEOMORPHISMS OF THE UNIT BALL

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Let $E^n = \{x: d(x, 0) \leqq 1, x \in R^n\}$, R^n is Euclidean n-space, $n \geqq 1$. If T and S are any two transformations defined on E^n to itself define $\|T-S\| = \sup\{d[T(x), S(x)]: x \in E^n\}$. Consider a continuous transformation T of E^n onto E^n. Denote by T^k the kth iterate of T, e.g. $T \circ T = T^2$. Suppose that $T \neq I$ (I the identity map) but T has $(n+1)$ fixed points on ∂E^n. J. Ax [1] has conjectured that under these conditions $\inf_k \|T^k - I\| > 0$.

Originally, the stronger conclusion $\|T^{k+1} - I\| \geqq \|T^k - I\|$ was expected to hold for all E^n, $n \geqq 1$; however, a counterexample was constructed in [2] for $n \geqq 2$. Note that for $n = 1$ this stronger conclusion implies $\inf_k \|T^k - I\| > 0$ and will be used later.

We first obtain a weaker result (Theorem A) for $n = 1$, 2 and then use this to prove the main result (Theorem B) for $n = 1$, 2.

A consequence of the main theorem is the known result that an infinite compact zero-dimensional group cannot act effectively on a two-dimensional manifold. The question is open for higher dimensions.

THEOREM A. *Let E^n be the unit ball in R^n, $n = 1$, 2 and let T be a continuous transformation of E^n onto E^n such that $T \neq I$, but restricted to ∂E^n, $T = I$. Then* $\inf_k \|T^k - I\| > 0$.

PROOF. For $n = 1$ see [2].

If T is many-one the theorem is trivially true; it is enough therefore to consider a homeomorphism of E^2 to E^2.

The rest of the proof is contrapositive. Given $\epsilon > 0$, $\exists N > 0$ such that $\|T^{k_i} - I\| < \epsilon$ for all $i > N$. Equivalently,

$$\sup\{d[T^{k_i}(x), x]; x \in E^n\} < \epsilon, \qquad i > N$$

which implies

$$d[T^{k_i}(x), x] < \epsilon \qquad \text{for all } x \in E^2, \quad i > N. \tag{1}$$

Let X be a metric space and consider a sequence of maps

$$g_1, g_2, \cdots, g_n, \cdots \tag{2}$$

of X into itself. The sequence (2) is called a *C-sequence* if for every

Received by the editors May 22, 1967.

[1] I wish to thank Professor Haskell Cohen for helpful discussions.

$\epsilon>0$, $\exists\delta(\epsilon)$, $N(\epsilon)$ such that $d(x, y)<\delta$ implies $d[g_n(x), g_n(y)]<\epsilon$ for all $n>N$. As an immediate consequence we note that if (2) is a C-sequence, and if each g_n is uniformly continuous then (2) is equi-uniformly continuous. A result of Edrei [**3**] now states that if h maps a totally bounded metric space onto itself such that some subsequence

$$h^{v_1}, h^{v_2}, \cdots, h^{v_m}, \cdots$$

of $h, h^2, h^3, \cdots, h^n, \cdots$ is a C-sequence then h is a homeomorphism and the set of maps h^n $[n=0, \pm 1, \pm 2, \cdots]$ is equi-uniformly continuous.

That $\{T^{k_i}\}_{i=1}^{\infty}$ is a C-sequence follows at once from

$$d[T^{k_i}(x), T^{k_i}(y)] \leqq d[T^{k_i}(x), x] + d[x, y] + d[T^{k_i}(y), y]$$

and (1). Hence $\{T^n\}$ is an equi-uniformly continuous set of transformations of E^2 onto E^2. This implies [**5**, p. 341] that T is almost periodic. A recent result by Foland [**4**, p. 1032] states that if $T|\partial E^2 = I$, and T is almost periodic then $T=I$. A contradiction.

THEOREM B. *Let E^n be the unit ball in R^n, $n=1$, 2 and let $T\neq I$ be a continuous transformation of E^n onto E^n such that T restricted to ∂E^n has $n+1$ fixed points. Then* $\inf_k \|T^k-I\|>0$.

PROOF. For $n=1$ Theorem B agrees precisely with Theorem A.

For $n=2$, suppose $\inf_k \|T^k-I\|=0$. We know that T has three fixed points, say x_1, x_2, x_3, on ∂E^2. Let x_1, x_2, x_3 be arranged in some order say clockwise on ∂E^2. Define $E_{13}=\{x: x\in\partial E^2$, and x lies between x_1 and $x_3\}$, where the subscripts indicate the points between which the elements of the set lie but the direction always being taken as clockwise. Since a homeomorphism of a compact set takes the boundary onto the boundary, T mapping E^2 onto E^2 clearly takes E_{13} onto E_{13}. $x_2\in E_{13}$ disallows reflection. T restricted to E_{13} therefore satisfies the conditions of Theorem A when $n=1$. Hence $T=I$ on E_{13}. Similarly $T=I$ on E_{31} and therefore T must be the identity on all ∂E^2. The conditions of Theorem A are now satisfied for $n=2$. This completes the proof.

REMARK. Let D^2 be any simply connected compact subset of R^2. Let $T: D^2\to D^2$ onto, continuous and such that $T\neq I$, $T|\partial D^2=I$. Then $\inf_k \|T^k-I\|>0$.

REMARK. Let D^2 be any simply connected compact subset of R^2, homeomorphic to E^2. Let $f: D^2\to D^2$ be onto, continuous, and such that f has three fixed points. Then $\inf_k \|f^k-I\|>0$.

Bibliography

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