INVERSES FOR FIBER SPACES

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1. **Introduction.** In [5, Proposition 1.2] M. Spivak has shown that spherical fiber spaces over finite complexes have inverses with respect to the operation of Whitney join. The main result of this paper, proved in §6, is a generalization of this theorem to finitely numerable spherical fiber spaces over arbitrary base spaces. We first prove an analogous theorem for fiber bundles with Euclidean space as fiber, using transition functions. We then show that enough of the theory of transition functions can be generalized to give essentially the same proof for fiber spaces. Our proof is more elementary than Spivak's in that we do not need the existence of a classifying space for spherical fiber spaces.

2. **Some notation.** E and B are topological spaces and $p\colon E\to B$ is a map. $\{U_i : i\in I\}$ is an open cover of B and $\{\rho_i : i\in I\}$ is a locally finite partition of unity such that supp $\rho_i=\{x:\rho_i(x)>0\}^-\subset U_i$. For each i, E_i is also a topological space and $p_i\colon E_i\to U_i$ is a map. If (E, p, B) is a fiber bundle (space) it is k-numerable provided $I=\{1, 2, \cdots, k\}$ and $E|U_i$ is trivial (fiber homotopically trivial) for $i=1, 2, \cdots, k$. Also $U_{ij}=U_i\cap U_j$ for $i, j\in I$.

3. **An inverse for a 2-numerable fiber bundle.**

THEOREM. *Let E be a 2-numerable fiber bundle with fiber $\boldsymbol{R}^n$ and with the group G_n of origin preserving homeomorphisms of $\boldsymbol{R}^n$ as structure group. Let $\phi_i\colon U_i\times\boldsymbol{R}^n\to E$ be coordinate charts, and let $g\colon U_{12}\to G_n$ be the transition function such that $\phi_2(b, x)=\phi_1(b, g(b)\cdot x)$. Let E^{-1} be the bundle formed in the same way, but using $g^{-1}\colon U_{12}\to G$, $g^{-1}(b)=g(b)^{-1}$, as transition function.*

Then the Whitney sum $E\oplus E^{-1}$ is trivial as an $\boldsymbol{R}^{2n}$ bundle.

PROOF. The transition function for $E\oplus E^{-1}$ is $g\times g^{-1}$, where $g\times g^{-1}(b)=g(b)\times g(b)^{-1}=(g(b)\times l_n)\circ(l_n\times g^{-1}(b))$. Here we consider $\boldsymbol{R}^{2n}$ as $\boldsymbol{R}^n\times\boldsymbol{R}^n$. Let e_i, $1\leqq i\leqq n$, be the standard basis vectors for $\boldsymbol{R}^n$. Let $\alpha_t\colon\boldsymbol{R}^n\to\boldsymbol{R}^n$ be the map which rotates each $e_i\times 0$ through an angle t towards $0\times e_i$ for $1\leqq i\leqq n$. Define

$$H(b, t) = \alpha_{t\pi/2}\circ(g(b)\times 1_n)\circ\alpha_{-t\pi/2}\circ(1_n\times g(b)^{-1}).$$

Then $H(b, 0)=g\times g^{-1}(b)$ and $H(b, 1)=1_n\times(g(b)\cdot g(b)^{-1})=1_{2n}$. H can

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be viewed as a transition function for a bundle over $B\times I$ which is trivial over $B\times 1$ and which can be identified with $E\oplus E^{-1}$ over $B\times 0$. By the covering homotopy theorem, $E\oplus E^{-1}$ is trivial.

4. **Pasting fiber spaces.** In order to imitate the proof just given we need to know that a 2-numerable fiber space can be characterized by a transition function. We state a more general theorem which allows us to put together any two fibrations if we are given a transition function, the result being unique up to fiber homotopy equivalence.

THEOREM. *Let* $p_i: E_i \to U_i$, $i=1, 2$, *be fiber spaces in the sense of Dold* [1, p. 238], *i.e. satisfying the weak covering homotopy property. Let* $\phi: E_1|U_{12} \to E_2|U_{12}$ *be a fiber homotopy equivalence. Then there is a fiber space* $p_\phi: E_\phi \to B = U_1 \cup U_2$ *and fiber homotopy equivalences* $\theta_i: E_i \to E_\phi | U_i$ *such that* θ_1 *and* $\theta_2 \circ \phi$ *are vertically homotopic over* U_{12}.

Moreover, if $p: E\to B$ *is a fiber space and* $\phi_i: E|U_i \to E_i$ *are fiber homotopy equivalences with inverses* $\bar\phi_i$, $i=1, 2$, *such that* $\phi_2 \circ \bar\phi_1$ *is vertically homotopic to* ϕ, *then there is a fiber homotopy equivalence* $\Phi: E \simeq E_\phi$ *such that* Φ *is vertically homotopic to* $\theta_i \circ \phi_i$ *over* U_i *for* $i=1, 2$.

REMARK 1. The E_i need not be numerable.

REMARK 2. This theorem is of independent interest. We are now working on a general theory of transition functions for fiber spaces based on this theorem.

PROOF. The construction of E_ϕ is due to Dold [2, Section 6.5]. We repeat the construction in order to prove the remainder of the theorem. Let

$$R = \{(x, w): x \in E_1, w: [0, 1] \to E_2, \quad p_2 \circ w([0, 1]) = p_1(x) \text{ and } w(1) = \phi(x)\}.$$

Let ϕ' be a homotopy inverse for ϕ and let ψ be a vertical homotopy from the identity to $\phi\circ\phi'$. Let ψ_y denote the path from y to $\phi\phi'(y)$ defined by ψ. Embed $E_1|U_{12}$ in R by $x\mapsto(x$, constant path at $\phi(x))$ and embed $E_2|U_{12}$ in R by $y\mapsto(\phi'(y), \psi_y)$. E_ϕ is the adjunction space formed by attaching E_1 and E_2 to R along these embeddings. The projection p_ϕ is the obvious one and the attaching defines maps $\theta_\phi: E_i\to E_\phi$. In the reference cited, Dold shows that each θ_i is a homeomorphism of E_i onto a fiberwise deformation retract of $E_\phi|U_i$ and concludes that E_ϕ is a fiber space.

We next define a vertical homotopy from θ_2 to $\theta_1\circ\phi'$ over U_{12}. Since ϕ' is an inverse to ϕ_1, it follows that θ_1 is vertically homotopic to $\theta_2\circ\phi$ over U_{12}. For $y\in E_2|U_{12}$ define

$$\alpha_{t,y}(s) = \psi_y(s+t) \qquad \text{for } 0 \leqq s \leqq 1-t,$$
$$= \phi\phi'(y) \qquad \text{for } 1-t \leqq s,$$

and set $K_t(y)=(\phi'(y),\ \alpha_{t,y})$. Then $K_0(y)=(\phi'(y),\ \psi_y)=\theta_2(y)$ and $K_1(y)=(\phi'(y),\ \alpha_{1,y})=(\phi'(y),$ constant path at $\phi\phi'(y))=\theta_1\circ\phi'(y)$.

Finally, we prove uniqueness. Assume E, ϕ_i and $\bar{\phi}_i$ given as described. Recall that $\{\rho_1, \rho_2\}$ is a partition of unity whose supports refine $\{U_1, U_2\}$. We have $\theta_1\circ\phi_1 \simeq \theta_2\circ\phi\circ\phi_1 \simeq \theta_2\circ\phi_2\circ\bar{\phi}_1\circ\phi_1 \simeq \theta_2\circ\phi_2$ over U_{12}. Let H be a vertical homotopy from $\theta_1\circ\phi_1$ to $\theta_2\circ\phi_2$ over U_{12}. Define Φ: $E\to E_\phi$ by

$$\Phi(x) = \theta_1\phi_1(x) \qquad \text{if } \rho_1(p(x)) = 1,$$
$$= H(\theta_1\phi_1(x),\ 1-\rho_1(p(x))) \qquad \text{if } p(x)\in \operatorname{supp}\rho_1,$$
$$= \theta_2\phi_2(x) \qquad \text{if } \rho_1(p(x)) = 0.$$

Φ is well defined and is clearly a fiber homotopy equivalence over U_1 and over U_2. By Dold [1, Theorem 3.3], Φ is a fiber homotopy equivalence.

COROLLARY. *If $E_i=U_i\times F$ and if ϕ is vertically homotopic to the identity, then E_ϕ is fiber homotopy trivial.*

PROOF. Apply uniqueness to $B\times F$.

5. **Inverses for 2-numerable spherical fiber spaces.** For spherical fiber spaces and the Whitney join we follow the notation of Spivak [5, Section 1] with the exception that fiber spaces need only satisfy the weak covering homotopy property. In particular, the Whitney join of two fiber spaces has the "small" topology of Milnor [3]. It is easy to verify directly that the join of two fiber spaces is again a fiber space.

THEOREM. *Every 2-numerable spherical fiber space has an inverse with respect to the operation of Whitney join.*

PROOF. Let $\phi_i\colon U_i\times S^{n-1}\to E| U_i$ be a fiber homotopy equivalence with inverse $\bar{\phi}_i$. Then $\phi=\bar{\phi}_1\circ\phi_2$: $U_{12}\times S^{n-1}\to U_{12}\times S^{n-1}$ is a transition function for E in the sense of §4. Let $\bar{\phi}=\bar{\phi}_2\circ\phi_1$. We contend that $E_\phi\oplus E_{\bar{\phi}}$ is fiber homotopy trivial. It is clear that $\phi\oplus\bar{\phi}$ is a transition function for $E_\phi\oplus E_{\bar{\phi}}$, and to complete the proof we need only the analogue of the homotopy H in §3. This is supplied by the following lemma which is undoubtedly well known.

LEMMA. *Let $H(S^{n-1})$ be the space of self-homotopy equivalences of S^{n-1} with the compact open topology. Let m, m': $H(S^{n-1})\times H(S^{n-1})$*

$\to H(S^{n-1} \oplus S^{n-1})$ be defined by $m(f, g) = f \oplus g$ and $m'(f, g) = 1 \oplus (f \circ g)$. Here $\oplus$ denotes join and 1 is the identity. Then m is homotopic to m'.

PROOF. Let ϕ: $S^{n-1} \oplus S^{n-1} \to S^{2n-1}$ be the standard homeomorphism, i.e. $\phi(x, t, y) = ((1-t)^2 + t^2)^{-1/2}((1-t)x, ty)$, where $x, y \in \mathbf{R}^n$ and $0 \leqq t \leqq 1$. Let $\rho_t = \phi^{-1} \circ \alpha_{t\pi/2} \circ \phi$ where $\alpha_{t\pi/2}$ is defined in §3. Finally, set $H_t(f, g) = \rho_t \circ (f \oplus 1) \circ \rho_{-t} \circ (1 \oplus g)$. This homotopy completes the proof of the lemma and the theorem.

6. Inverses for finitely numerable fiber spaces or bundles.

In the following theorem fiber spaces may be replaced by fiber bundles as in §3.

THEOREM. *Let p: $E \to B$ be a k-numerable spherical fiber space for some integer k. Then E has an inverse with respect to the Whitney join operation.*

PROOF. This consists in verifying that we can perform the steps in Milnor's proof [4, Theorem 3] that microbundles over finite complexes have inverses. Having just settled the case $k = 2$, we assume that E is $(k+1)$-numerable. Let $V_1 = U_1 \cup \cdots \cup U_k$ and $V_2 = U_{k+1}$. Let $\tilde{E}$ be an inverse for $E_k = E | V_1$. Let ϵ be a trivial spherical fiber space of the same dimension as E. Then $\tilde{E} \oplus E$ is fiber homotopy equivalent over $V_2 \cap V_1$ to $\tilde{E} \oplus E_k$; hence $\tilde{E} \oplus \epsilon$ is trivial over $V_2 \cap V_1$. By the pasting theorem, there is a fiber space E' over $V_1 \cup V_2$ which is equivalent to $\tilde{E}$ over V_1 and to ϵ over V_2. $E \oplus E'$ is then trivial over V_1 and over V_2 and therefore has an inverse E''. An inverse for E is $E' \oplus E''$.

ADDED IN PROOF. After this paper was submitted, there appeared a similar proof for the fiber bundle case in J. Kister, *Inverses of Euclidean bundles*, Michigan Math. J. **14** (1967), 349–352.

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