1. Introduction. In [5, Proposition 1.2] M. Spivak has shown that spherical fiber spaces over finite complexes have inverses with respect to the operation of Whitney join. The main result of this paper, proved in §6, is a generalization of this theorem to finitely numerable spherical fiber spaces over arbitrary base spaces. We first prove an analogous theorem for fiber bundles with Euclidean space as fiber, using transition functions. We then show that enough of the theory of transition functions can be generalized to give essentially the same proof for fiber spaces. Our proof is more elementary than Spivak's in that we do not need the existence of a classifying space for spherical fiber spaces.

2. Some notation. \( E \) and \( B \) are topological spaces and \( p: E \rightarrow B \) is a map. \( \{ U_i : i \in I \} \) is an open cover of \( B \) and \( \{ \rho_i : i \in I \} \) is a locally finite partition of unity such that \( \text{supp} \rho_i = \{ x : \rho_i(x) > 0 \} \subset U_i \). For each \( i \), \( E_i \) is also a topological space and \( \rho_i : E_i \rightarrow U_i \) is a map. If \( (E, \rho, B) \) is a fiber bundle (space) it is \( k \)-numerable provided \( I = \{ 1, 2, \ldots, k \} \) and \( \rho_i \mid U_i \) is trivial (fiber homotopically trivial) for \( i = 1, 2, \ldots, k \). Also \( U_{ij} = U_i \cap U_j \) for \( i, j \in I \).

3. An inverse for a 2-numerable fiber bundle.

Theorem. Let \( E \) be a 2-numerable fiber bundle with fiber \( \mathbb{R}^n \) and with the group \( G_n \) of origin preserving homeomorphisms of \( \mathbb{R}^n \) as structure group. Let \( \phi_i : U_i \times \mathbb{R}^n \rightarrow E \) be coordinate charts, and let \( g : U_{12} \rightarrow G_n \) be the transition function such that \( \phi_2(b, x) = \phi_1(b, g(b) \cdot x) \). Let \( E^{-1} \) be the bundle formed in the same way, but using \( g^{-1} : U_{12} \rightarrow G_n \) where \( g^{-1} = g(b)^{-1} \), as transition function.

Then the Whitney sum \( E \oplus E^{-1} \) is trivial as an \( \mathbb{R}^{2n} \) bundle.

Proof. The transition function for \( E \oplus E^{-1} \) is \( g \times g^{-1} \), where \( g \times g^{-1} (b) = g(b) \times g(b)^{-1} = (g(b) \times l_n) \circ (l_n \times g^{-1}(b)) \). Here we consider \( \mathbb{R}^{2n} \) as \( \mathbb{R}^n \times \mathbb{R}^n \). Let \( e_i \), \( 1 \leq i \leq n \), be the standard basis vectors for \( \mathbb{R}^n \). Let \( \alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the map which rotates each \( e_i \times 0 \) through an angle \( t \) towards \( 0 \times e_i \) for \( 1 \leq i \leq n \). Define

\[
H(b, t) = \alpha_{t \pi/2} \circ (g(b) \times 1_n) \circ \alpha_{-t \pi/2} \circ (1_n \times g(b)^{-1}).
\]

Then \( H(b, 0) = g \times g^{-1}(b) \) and \( H(b, 1) = 1_n \times (g(b) \cdot g(b)^{-1}) = 1_{2n} \). \( H \) can
be viewed as a transition function for a bundle over $B \times I$ which is trivial over $B \times 1$ and which can be identified with $E \oplus E^{-1}$ over $B \times 0$. By the covering homotopy theorem, $E \oplus E^{-1}$ is trivial.

4. Pasting fiber spaces. In order to imitate the proof just given we need to know that a 2-numerable fiber space can be characterized by a transition function. We state a more general theorem which allows us to put together any two fibrations if we are given a transition function, the result being unique up to fiber homotopy equivalence.

**Theorem.** Let $p_i: E_i \rightarrow U_i$, $i = 1, 2$, be fiber spaces in the sense of Dold [1, p. 238], i.e. satisfying the weak covering homotopy property. Let $\phi: E_1 \mid U_{12} \rightarrow E_2 \mid U_{12}$ be a fiber homotopy equivalence. Then there is a fiber space $p_4: E_4 \rightarrow B = U_1 \cup U_2$ and fiber homotopy equivalences $\theta_i: E_i \rightarrow E_4 \mid U_i$ such that $\theta_1$ and $\theta_2 \circ \phi$ are vertically homotopic over $U_{12}$.

Moreover, if $p: E \rightarrow B$ is a fiber space and $\phi_i: E \mid U_i \rightarrow E_i$ are fiber homotopy equivalences with inverses $\phi_i^-$, $i = 1, 2$, such that $\phi_2 \circ \phi_1$ is vertically homotopic to $\phi$, then there is a fiber homotopy equivalence $\Phi: E \simeq E_4$ such that $\Phi$ is vertically homotopic to $\theta_i \circ \phi_i$ over $U_i$ for $i = 1, 2$.

**Remark 1.** The $E_i$ need not be numerable.

**Remark 2.** This theorem is of independent interest. We are now working on a general theory of transition functions for fiber spaces based on this theorem.

**Proof.** The construction of $E_4$ is due to Dold [2, Section 6.5]. We repeat the construction in order to prove the remainder of the theorem. Let

$$R = \{(x, w): x \in E_1, w: [0, 1] \rightarrow E_2, \quad p_2 \circ w([0, 1]) = p_1(x) \text{ and } w(1) = \phi(x)\}.$$

Let $\phi'$ be a homotopy inverse for $\phi$ and let $\psi$ be a vertical homotopy from the identity to $\phi \circ \phi'$. Let $\psi_y$ denote the path from $y$ to $\phi \phi'(y)$ defined by $\psi$. Embed $E_1 \mid U_{12}$ in $R$ by $x \mapsto (x$, constant path at $\phi(x)$) and embed $E_2 \mid U_{12}$ in $R$ by $y \mapsto (\phi'(y), \psi_y)$. $E_4$ is the adjunction space formed by attaching $E_1$ and $E_2$ to $R$ along these embeddings. The projection $p_4$ is the obvious one and the attaching defines maps $\theta_4: E_i \rightarrow E_4$. In the reference cited, Dold shows that each $\theta_i$ is a homeomorphism of $E_i$ onto a fiberwise deformation retract of $E_4 \mid U_i$ and concludes that $E_4$ is a fiber space.

We next define a vertical homotopy from $\theta_2$ to $\theta_1 \circ \phi'$ over $U_{12}$. Since $\phi'$ is an inverse to $\phi_1$, it follows that $\theta_1$ is vertically homotopic to $\theta_2 \circ \phi$ over $U_{12}$. For $y \in E_2 \mid U_{12}$ define
\[ \alpha_{1,v}(s) = \psi_v(s + t) \quad \text{for } 0 \leq s \leq 1 - t, \]
\[ = \phi \phi'(y) \quad \text{for } 1 - t \leq s, \]

and set \( K_1(y) = (\phi'(y), \alpha_{1,v}) \). Then \( K_0(y) = (\phi'(y), \psi_v) = \theta_0(y) \) and \( K_1(y) = (\phi'(y), \alpha_{1,v}) = (\phi'(y), \text{constant path at } \phi \phi'(y)) = \theta_1 \circ \phi'(y) \).

Finally, we prove uniqueness. Assume \( E, \phi_i \) and \( \phi_i \) given as described. Recall that \( \{ \rho_1, \rho_2 \} \) is a partition of unity whose supports refine \( \{ U_1, U_2 \} \). We have \( \theta_1 \circ \phi_1 \sim \theta_2 \circ \phi_2 \circ \phi_1 \circ \phi_2 \circ \theta_2 \circ \phi_2 \) over \( U_{12} \). Let \( H \) be a vertical homotopy from \( \theta_1 \circ \phi_1 \) to \( \theta_2 \circ \phi_2 \) over \( U_{12} \). Define \( \Phi : E \to E_\phi \) by

\[ \Phi(x) = \theta_1 \phi_1(x) \quad \text{if } \rho_1(\rho(x)) = 1, \]
\[ = H(\theta_1 \phi_1(x), 1 - \rho_1(\rho(x))) \quad \text{if } \rho(x) \in \text{supp } \rho_1, \]
\[ = \theta_2 \phi_2(x) \quad \text{if } \rho_1(\rho(x)) = 0. \]

\( \Phi \) is well defined and is clearly a fiber homotopy equivalence over \( U_1 \) and over \( U_2 \). By Dold [1, Theorem 3.3], \( \Phi \) is a fiber homotopy equivalence.

**Corollary.** If \( E_\phi = U_1 \times F \) and if \( \phi \) is vertically homotopic to the identity, then \( E_\phi \) is fiber homotopy trivial.

**Proof.** Apply uniqueness to \( B \times F \).

5. Inverses for 2-numerable spherical fiber spaces. For spherical fiber spaces and the Whitney join we follow the notation of Spivak [5, Section 1] with the exception that fiber spaces need only satisfy the weak covering homotopy property. In particular, the Whitney join of two fiber spaces has the "small" topology of Milnor [3]. It is easy to verify directly that the join of two fiber spaces is again a fiber space.

**Theorem.** Every 2-numerable spherical fiber space has an inverse with respect to the operation of Whitney join.

**Proof.** Let \( \phi_i : U_1 \times S^{n-1} \to E \mid U_1 \) be a fiber homotopy equivalence with inverse \( \phi_i \). Then \( \phi = \phi_1 \circ \phi_2 : U_{12} \times S^{n-1} \to U_{12} \times S^{n-1} \) is a transition function for \( E \) in the sense of §4. Let \( \phi' = \phi_2 \circ \phi_1 \). We contend that \( E_\phi \oplus E' \phi \) is fiber homotopy trivial. It is clear that \( \phi \oplus \phi' \) is a transition function for \( E_\phi \oplus E' \phi \), and to complete the proof we need only the analogue of the homotopy \( H \) in §3. This is supplied by the following lemma which is undoubtedly well known.

**Lemma.** Let \( H(S^{n-1}) \) be the space of self-homotopy equivalences of \( S^{n-1} \) with the compact open topology. Let \( m, m' : H(S^{n-1}) \times H(S^{n-1}) \)
→H(S^n−1⊕S^n−1) be defined by m(f, g) = f ⊕ g and m'(f, g) = 1 ⊕ (f ◦ g).
Here ⊕ denotes join and 1 is the identity. Then m is homotopic to m'.

Proof. Let φ: S^n−1 ⊕ S^n−1 → S^{2n−1} be the standard homeomorphism,
i.e. φ(x, t, y) = ((1−t)^2+t^2)^{-1/2}((1−t)x, ty), where x, y ∈ R^n and
0 ≤ t ≤ 1. Let ρ_t = φ^{-1} ◦ α_{t/2} ◦ φ where α_{t/2} is defined in §3. Finally,
set H_t(f, g) = ρ_t ◦ (f ⊕ 1) ◦ ρ_{t} ◦ (1 ⊕ g). This homotopy completes
the proof of the lemma and the theorem.

6. Inverses for finitely numerable fiber spaces or bundles. In the
following theorem fiber spaces may be replaced by fiber bundles as
in §3.

Theorem. Let p: E→B be a k-numerable spherical fiber space for
some integer k. Then E has an inverse with respect to the Whitney join
operation.

Proof. This consists in verifying that we can perform the steps
in Milnor's proof [4, Theorem 3] that microbundles over finite com-
plexes have inverses. Having just settled the case k = 2, we assume
that E is (k + 1)-numerable. Let V_1 = U_1 ∪ ⋯ ∪ U_k and V_2 = U_{k+1}.
Let E be an inverse for E_k = E | V_1. Let ε be a trivial spherical fiber
space of the same dimension as E. Then E ⊕ E is fiber homotopy
equivalent over V_2 ∩ V_1 to E ⊕ E_k; hence E ⊕ ε is trivial over V_2 ∩ V_1.
By the pasting theorem, there is a fiber space E' over V_1 ∪ V_2 which
is equivalent to E over V_1 and to ε over V_2. E ⊕ E' is then trivial over
V_1 and over V_2 and therefore has an inverse E''. An inverse for E is
E' ⊕ E''.

Added in proof. After this paper was submitted, there appeared
a similar proof for the fiber bundle case in J. Kister, Inverses of

Bibliography

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