EXTENSION OF COHERENT ANALYTIC SUBSHEAVES

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In this note we prove the following result.

THEOREM. *Suppose* $\mathfrak{I}$ *is a coherent analytic sheaf on a Stein space* $(X, \mathcal{H})$ *in the sense of Grauert* [2, §1] *and* $\mathcal{S}$ *is a coherent analytic subsheaf of* $\mathfrak{I}|U$ *for some open neighborhood* U *of the boundary* ∂X *of* X. *If for every* $x \in U$, $\mathcal{S}_x$, *as a* $\mathcal{H}_x$*-submodule of* $\mathfrak{I}_x$, *has no associated prime ideal of dimension* $\leqq 1$, *then there exists a coherent analytic subsheaf* $\mathcal{S}^*$ *of* $\mathfrak{I}$ *on* $(X, \mathcal{H})$ *such that* $\mathcal{S}^*$ *agrees with* $\mathcal{S}$ *on some open neighborhood of* ∂X.

This theorem extends [3, Chapter VII.D.6].

NOTATIONS. codh denotes homological codimension. $D(\mathfrak{I})$ denotes $\{x \in X \mid \text{codh}_{\mathcal{H}_x} \mathfrak{I}_x < 3\}$. For $r > 0$, $R_r = \{z \in \mathbf{C}^n \mid (\sum_{i=1}^n |z_i|^2)^{1/2} < r\}$. For $s > r > 0$, $R_{r,s} = R_s - R_r^-$, where $^-$ denotes topological closure.

LEMMA. *Suppose* $\mathfrak{M} \subset \mathcal{O}(G)^p$ *is a coherent subsheaf, where* $\mathcal{O}(G)$ *is the structure sheaf of an open subset* G *of* $\mathbf{C}^n$ $(n \geqq 3)$ *and*

(*) $\mathfrak{M}_x$ *as an* $\mathcal{O}(G)_x$*-submodule of* $\mathcal{O}(G)_x^p$ *has no associated prime ideal of* dim $\leqq 1$ *for every* $x \in G$.

Then $D(\mathfrak{M})$ *is either discrete or empty.*

PROOF. Suppose not. Since $D(\mathfrak{M})$ is a subvariety in G [5, Satz 5] there is an irreducible 1-dimensional subvariety Z in a connected Stein open subset H of G such that $Z \subset D(\mathfrak{M})$. Take a holomorphic function $f \not\equiv 0$ on H vanishing on Z. Take $x \in Z$. $f_x \mathfrak{M}_x$ as an $\mathcal{O}(G)_x$-submodule of $\mathfrak{M}_x$ has no associated prime ideal of dimension $\leqq 1$, for otherwise there is a prime ideal P in $\mathcal{O}(G)_x$ of dim $\leqq 1$ and $s \in \mathfrak{M}_x$ such that $sP^k \subset f_x \mathfrak{M}_x$ for some k and $s \notin f_x \mathfrak{M}_x$. The meromorphic function-germ sf_x^{-1} is holomorphic, because it is holomorphic outside a subvariety-germ of codim $\geqq 2$. $sf_x^{-1}P^k \subset \mathfrak{M}_x$ and $sf_x^{-1} \notin \mathfrak{M}_x$. (*) is contradicted. Take a holomorphic function g defined in some open neighborhood W of x in H and vanishing on $Z \cap W$ such that g_x does not belong to any associated prime ideal of $f_x \mathfrak{M}_x$ as an $\mathcal{O}(G)_x$-submodule of $\mathfrak{M}_x$. g_x is not a zero-divisor for $\mathfrak{M}_x / f_x \mathfrak{M}_x$. By coherence of the kernel of the sheaf-homomorphism $\mathfrak{M}/f\mathfrak{M} \to \mathfrak{M}/f\mathfrak{M}$ defined by multiplication by g, after shrinking of W we can assume that g_y is not a zero-divisor for $\mathfrak{M}_y / f_y \mathfrak{M}_y$ for $y \in W$. Since $Y = \{y \in W \mid \text{codh } (\mathfrak{M}/(f\mathfrak{M} + g\mathfrak{M}))_y \leqq 0\}$

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is at most zero-dimensional [5, Satz 5], $\exists z \in (Z \cap W) - Y$. Then $\operatorname{codh} \mathfrak{M}_z \geqq 3$. Contradiction, q.e.d.

PROOF OF THEOREM. $K = X - U$ is compact. By replacing X by a relatively compact Stein neighborhood of K, we can assume w.l.o.g. that X is a complex subspace of $\boldsymbol{C}^n$ with $n \geqq 3$ (Einbettungssatz, [6]) and we have a sheaf-epimorphism h: $\mathcal{O}^p \rightarrow \tilde{\mathfrak{J}}$, where $\tilde{\mathfrak{J}}$ is the trivial extension of $\mathfrak{J}$ on $\boldsymbol{C}^n$ and $\mathcal{O}$ is the structure sheaf of $\boldsymbol{C}^n$. By replacing X by $\boldsymbol{C}^n$ and $\mathcal{S}$ by $h^{-1}(\tilde{\mathcal{S}})$, where $\tilde{\mathcal{S}}$ is the trivial extension of $\mathcal{S}$ on $\boldsymbol{C}^n - K$, we can assume w.l.o.g. that $X = \boldsymbol{C}^n$ and $\mathfrak{J} = \mathcal{O}^p$. By the lemma we can choose $s > r > d > 0$ such that $R_{r-d} \supset K$ and $D(\mathcal{S}) \cap R_{r-d,s+d} = \varnothing$. For some $0 < a$, $b < d$, the restriction map $H^1(R_{r-a,s+b}, \mathcal{S}) \rightarrow H^1(R_{r,s}, \mathcal{S})$ is surjective [1, Propositions 16 and 17, §17]. $\dim_C H^1(R_{r,s}, \mathcal{S}) < \infty$ (cf. the proof of Theorem 11, [1, §17]). Take $x \in R_{r,s}$ and a complex-linear function f on $\boldsymbol{C}^n$ such that $f(x) = 0$ and the set V of zeroes of f is disjoint from R_r^-. The exact sequence $0 \rightarrow \mathcal{S} \xrightarrow{u} \mathcal{S} \rightarrow \mathcal{S}/f\mathcal{S} \rightarrow 0$, where u is defined by multiplication by f, yields the exact sequence $\Gamma(R_{r,s}, \mathcal{S}) \xrightarrow{v} \Gamma(R_{r,s}, \mathcal{S}/f\mathcal{S}) \rightarrow H^1(R_{r,s}, \mathcal{S}) \rightarrow H^1(R_{r,s}, \mathcal{S}) \rightarrow H^1(R_{r,s}, \mathcal{S}/f\mathcal{S})$. Since $V \cap R_{r,s}$ is Stein, $H^1(R_{r,s}, \mathcal{S}/f\mathcal{S}) = 0$. $\dim_C H^1(R_{r,s}, \mathcal{S}) < \infty$ implies that v is surjective. Let m be the maximal ideal of $\mathcal{O}_x$. w: $\Gamma(R_{r,s}, \mathcal{S}/f\mathcal{S}) \rightarrow \mathcal{S}_x/m\mathcal{S}_x$ is surjective, because $V \cap R_{r,s}$ is Stein. $w \circ v$ is surjective. By Krull-Azumaya Lemma [4, (4.1)], $\Gamma(R_{r,s}, \mathcal{S})$ generates $\mathcal{S}_x$. Since x is arbitrary, $\mathcal{S}$ restricted to $R_{r,s}$ is generated by sections on $R_{r,s}$. Extensions of elements of $\Gamma(R_{r,s}, \mathcal{S})$ form a subset S of $\Gamma(R_s, \mathcal{O}^p)$. S generates a coherent subsheaf $\mathfrak{F}$ of $\mathfrak{J}$ on R_s. Define $\mathcal{S}^*$ to be $\mathfrak{F}$ on R_s and to be $\mathcal{S}$ on $\boldsymbol{C}^n - R_r^-$. Then $\mathcal{S}^*$ is the required extension, q.e.d.

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