INTERSECTIONS OF MAXIMAL STARSHAPED SETS

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0. **Introduction.** In Valentine [1, p. 183] the problem of characterizing starshaped sets in terms of maximal convex sets was posed. One published solution says that the convex kernel of a set is the intersection of all the maximal convex subsets of the set [2, p. 280]. In this paper we investigate the analogous problem of describing the intersection of all maximal starshaped subsets of a set. A *maximal starshaped subset* X of a set Y is a starshaped subset of Y which is not properly contained in any other starshaped subset of Y. Since the property of being starshaped is not an intersectional property, it seems unlikely that the intersection of maximal starshaped subsets of a given set would be starshaped. Indeed, the following example shows the situation to be even more complex than merely absence of the intersectional property.

Let $T_n = \{(x, y) \mid n-1 \leqq y \leqq n,\ n-x \leqq y\}$, and $S_n = \bigcup_{i=1}^{n} T_i$; then S_n is starshaped with convex kernel, $\mathrm{ck}(S_n)$, equal to $K_n = \{(x, y) \mid 0 \leqq y \leqq 1, n-x \leqq y\}$. If $S = \bigcup_{n=1}^{\infty} S_n$, then $\mathrm{ck}(S) \subset \bigcup_{n=1}^{\infty} \mathrm{ck}(S_n) = \emptyset$. Thus S is not starshaped even though it is the union of an ascending chain of starshaped sets. Furthermore, S has no maximal starshaped subsets. If $M \subset S$ were a maximal starshaped subset, then there would be at least one point $(x, y) \in \mathrm{ck}(M)$. In fact M would be precisely the set of points that (x, y) sees via S. However, the point $(x+1, y)$ sees every point which (x, y) does, and more. Thus M is not maximal.

In contrast with the preceding example, it is shown in §1 that compact subsets of Euclidean space, E^n, have maximal starshaped subsets. In §2, it is shown that the intersection of the maximal starshaped subsets in a suitably restricted setting is starshaped.

1. **Existence of maximal starshaped sets.** Let S be a compact set in E^n and let $\mathfrak{F}$ denote the family of all classes C of maximal convex subsets of S for which $\bigcap C \neq \emptyset$. Observe that a maximal convex subset of S is compact. We note two properties of $\mathfrak{F}$. First, if D is a finite subclass of some $C \in \mathfrak{F}$ then $\emptyset \neq \bigcap C \subset \bigcap D$, so $D \in \mathfrak{F}$. Also, if C is a class of sets such that each finite subclass is in $\mathfrak{F}$, then $C \in \mathfrak{F}$ by compactness and the definition of $\mathfrak{F}$. Thus $\mathfrak{F}$ is a family of finite character.

THEOREM 1.1. *There exists a maximal starshaped subset T of any compact set S in E^n and every maximal starshaped subset is closed.*

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PROOF. By the preceding, Tukey's Lemma gives a maximal class C of maximal convex subsets of S for which $\bigcap C \neq \emptyset$. Let $T = \bigcup C$; by using the fact that a starshaped set is the union of its maximal convex subsets, we see that T is indeed a maximal starshaped subset of S. Noting that the closure, T^-, is starshaped and $T^- \subset S$, we see that $T = T^-$. That is, T is closed.

COROLLARY 1.2. *If $S \subset E^n$ is compact and B is any starshaped subset of S, then there exists a maximal starshaped subset $T \subset S$ such that $B \subset T$.*

PROOF. Express B as the union of its maximal convex subsets; then let $T = \bigcup C$, where C is one maximal class of maximal convex subsets of S with $\bigcap C \neq \emptyset$, at least one member containing each one of the maximal convex subsets of B.

2. **Intersections of maximal starshaped sets in the plane.**

Hereafter S is always taken to be a compact simply connected set in the plane. Likewise S_α, α in an index set I, will represent a maximal starshaped subset of S; and A is taken to be the intersection of all the maximal starshaped subsets of S, i.e. $A = \bigcap_{\alpha \in I} S_\alpha$. We note that A, perhaps empty, is closed and thus compact.

Particular notations are as follows: pq denotes the closed segment established by the points p and q; Δpqr denotes the convex hull of the three points, p, q, and r; $L(p, q)$ is the line established by the points p and q; and ${}_kC_{pq}$ denotes the cone opposite p and q with vertex k, i.e. ${}_kC_{pq} = \{x \mid x = \lambda p + \mu q + \nu k,\ \lambda + \mu + \nu = 1,\ \lambda \leqq 0,\ \mu \leqq 0\}$.

LEMMA 2.1. *If $p, q \in A$, then $pq \subset A$ if and only if $pq \subset S$.*

PROOF. The "only if" part is immediate. If $pq \subset S$ and $k \in \operatorname{ck}(S_\alpha)$, then we have $pq \cup qk \cup kp \subset S$. So $\Delta pqk \subset S$. This means that k sees all of Δpqk, so $S_\alpha \cup \Delta pqk$ is a starshaped subset of S having k in its kernel. Consequently we have $pq \subset S_\alpha$ by the maximality of S_α. That is $pq \subset A$.

Observe that simple connectedness and a standard sequence argument gives the following. If $p, q \in A$ and $pq \not\subset A$, then the set of points, B, that see p and q via S is a compact set contained in one of the open half planes of $L(p, q)$.

LEMMA 2.2. *The set B, as given above, contains a unique element which is closest to $L(p, q)$.*

PROOF. Since the distance from points of B to $L(p, q)$ is a positive continuous function defined on a compact set, we observe that a closest point exists. Distinct closest points x and y in B establish

$L(x, y)$ parallel to $L(p, q)$. Adjusted notation, if necessary, gives $xq \cap yp$ to be a point of B closer than the minimum distance.

LEMMA 2.3. *Every pair of points of A can be joined in A by a polygonal path of no more than two edges.*

PROOF. If $p, q \in A$ and $pq \not\subset A$, let m be the unique point of Lemma 2.2. If $k \in \mathrm{ck}(S_\alpha)$, we have $k \in {}_mC_{pq}$. But simple connectedness of S gives the quadrilateral $kpmq$ and its interior to be a subset of S. Now k sees m, so $pm \cup mq \subset S_\alpha$. Since α was arbitrary, it follows that $pm \cup mq \subset A$.

THEOREM 2.4. *The intersection of the maximal starshaped subsets of a compact, simply connected set in E_2 is starshaped or empty.*

PROOF. Let p, q, r be three points of A such that no point of A sees all three points via A. Otherwise Krasnoselskii's Theorem says that A is starshaped [1].

For the first case assume that p, q, and r are collinear with q between p and r.

By our initial assumption $pr \not\subset A$; suppose $qr \not\subset A$ and $pq \subset A$. Then Lemma 2.2 establishes a point m closest to $L(q, r)$ and Theorem 2.3 yields $qm \cup mr \subset A$. As before $\mathrm{ck}(S_\alpha) \subset {}_mC_{qr}$ for any α. If $k \in \mathrm{ck}(S_\alpha)$, we have $pk \cup kq \cup rk \subset S$. Simple connectedness gives $pm \subset S$. Thus, Lemma 2.1 says $pm \subset A$ and the resulting contradiction—$pm \cup mq \cup mr \subset A$—assures that $pq \not\subset A$. Similarly $qr \not\subset A$.

Let us now assume that none of the segments between p, q, and r is contained in A. Again Lemma 2.2 establishes a point m closest to $L(p, r)$ for the points p and r with $pm \cup mr \subset A$. Select $k \in \mathrm{ck}(S_\alpha)$ and note that $k \in {}_mC_{pr}$. If $m \in kq$, we have a contradiction, so assume that $kq \cap (rm \cup pm)$ is a point distinct from m. Without loss of generality let the point of intersection be on rm. Now apply Lemma 2.2 to establish a point n closest to $L(p, q)$. The point n must be such that $k \in {}_nC_{pq}$, i.e. $n \in \Delta pkq$. Extend qm to intersect pk in a point j. If $n \in \Delta pqj$, we observe that $mq \subset A$, a contradiction. Otherwise, either $nr \subset A$ or nq extended to intersect mr yields a point of A that sees p, q and r via A.

We now observe that all the above excludes the possibility of p, q and r being collinear.

Since p, q and r are not collinear, we employ them as a barycentric basis to describe regions of the plane. For example, a $(+, -, 0)$ point k is such that $k = \alpha p + \beta q + \gamma r$ with $\alpha + \beta + \gamma = 1$ and $\alpha > 0$, $\beta < 0$, $\gamma = 0$.

Suppose $k \in \mathrm{ck}(S_\alpha)$; two cases are trivial—namely $(0, 0, +)$ and $(0, +, +)$. Particular permutations of these sign symbols are assumed without loss of generality. The three cases $(0, -, +)$, $(+, -, +)$ and $(-, +, -)$ are disposed of simultaneously (i) with a proof identical in wording to the paragraph that dispenses with p, q and r being collinear and none of their segments in A, and (ii) with minor modifications for the cases in which one of the segments pq, qr, pr is in A.

The final possibility is for $k \in \mathrm{ck}(S_\alpha)$ to be a $(+, +, +)$-point. Here Lemma 2.2 and Theorem 2.3 assure us that $kp \cup kq \cup kr$ is contained in a "three-pointed star region" all of whose edges are segments of A. Simple connectedness and Lemma 2.1 ensure that this region (and, in particular $kp \cup kq \cup kr$) is a subset of A.

References

1. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

2. F. A. Toranzos, *Radial functions of convex and starshaped bodies*, Amer. Math. Monthly **74** (1967), 278–280.

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