A COMPLETE R_Λ-HARMONICALLY IMMERSED SURFACE IN E^3 ON WHICH $H \neq 0$

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Suppose S is an oriented surface harmonically immersed in E^3 with respect to the conformal structure R_Λ determined upon it by some fixed positive definite linear combination

$$\Lambda = fI + gII$$

of the fundamental forms I and II, with f and g smooth, real valued functions. Where mean curvature H vanishes, Λ must be proportional to I (see Lemma 6 of [1]), so that $R_\Lambda = R_I$, and S is harmonically immersed with respect to ordinary conformal structure. But where H does not vanish, Gauss curvature K is negative, while Λ must be proportional to the positive definite form

$$II' = \frac{1}{(H^2 - K)^{1/2}} (HII - KI),$$

so that $R_\Lambda = R_{II'}$ (see Lemmas 2 and 6 of [1]). In fact, unless $R_\Lambda \equiv R_I$ on S, the points at which $R_\Lambda = R_I$ (and $H = 0$) must be isolated.

We proved in [1] that H cannot be bounded away from zero on a complete R_Λ-harmonically immersed surface S in E^3, and asked (in effect) whether there could be a complete R_Λ-harmonically immersed surface S in E^3 on which H never vanishes. In this paper we establish the existence of a complete $R_{II'}$-harmonically immersed surface S in E^3 on which H is everywhere negative.

In fact, the example used in [1] to verify the existence of $R_{II'}$-harmonically imbedded surface patches on which $H \neq 0$ can be adapted to this purpose. Thus, take for S the finite u, v-plane provided with the forms

$$I = ((e^u + 2)/2)du^2 + (e^u/2)dv^2$$

and

$$II = \tfrac{1}{2}(e^u/(e^u + 2))^{1/2}(du^2 - dv^2).$$

Lemma 1 of [1] may be used to check that any immersion of S with I and II as fundamental forms is harmonic with respect to the conformal structure of the u, v-plane. On the other hand, Lemma 4 of

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[2] establishes that the conformal structure of the u, v-plane will be $R_{II'}$ structure on the surface so immersed. Finally, mean curvature H associated with I and II is given by

$$H = -(1/(e^u + 2)^3 e^u)^{1/2},$$

which makes H everywhere negative as claimed. To show that S may be globally immersed in E^3 with I and II as fundamental forms, we note that I is positive definite and that the Codazzi-Mainardi and theorem-egregium equations are satisfied. We then apply the following elementary observation.

FACT. *Let D be any simply connected open subset of the plane E^2 provided with smooth quadratic forms I (positive definite) and II which satisfy the theorem egregium and Codazzi-Mainardi equations. Then there exists an immersion of D in E^3 achieving I and II as fundamental forms (and uniquely determined up to motions of E^3).*

PROOF (OUTLINED). The fundamental theorem of surface theory guarantees that some neighborhood of any $p \in D$ may be imbedded in E^3 with I and II as fundamental forms, and uniquely so up to motions of E^3. Consider first the case in which $D \subset E^2$ is defined by

$$D = \{x, y \mid x^2 + y^2 < R\},$$

where $0 < R \leq \infty$. Let $R_0 > 0$ be the largest real number such that

$$D_0 = \{x, y \mid x^2 + y^2 < R_0\}$$

may be immersed in E^3 with the given I and II as fundamental forms. Let $X_0: D_0 \to E^3$ be such an immersion. If $R_0 < R$, any $p \in \partial D_0$ is the center of a disk Δ which may be imbedded in E^3 with I and II as fundamental forms, and with the same values as X_0 on $D_0 \cap \Delta$. Covering ∂D_0 by a finite number of such disks, one checks easily that the (uniquely determined) imbeddings involved coincide wherever the disks intersect. Extending X_0 by these imbeddings, one has a contradiction to the definition of R_0, unless $R_0 = R$. In case D is neither a disk nor E^2 itself, the Riemann mapping theorem may be applied, since D is simply connected. Thus there is a C^ω diffeomorphism f of D onto the unit disk. Transforming I and II by the rules appropriate to quadratic forms, we are back to the case already discussed, with $R = 1$. Composing f with the immersion of the unit disk, the required immersion of D is achieved.

It remains to check that I is complete on S. Let γ be any smooth divergent arc on S, so that γ is contained in no compact subset of S, while

$$(*) \qquad \int_\gamma (du^2 + dv^2)^{1/2} = \infty$$

since the ordinary Euclidean metric on S is complete. We must show that the length $l(\gamma)$ assigned to γ by the metric I is infinite. Suppose first that for some real constant c, $u \geqq c$ everywhere on γ. Then, using (*),

$$l(\gamma) = \int \left[\left(\frac{e^u + 2}{2} \right) du^2 + \left(\frac{e^u}{2} \right) dv^2 \right]^{1/2}$$
$$\geqq (e^c/2)^{1/2} \int_\gamma (du^2 + dv^2)^{1/2} = \infty .$$

If, on the other hand, u assumes arbitrarily large negative values, then

$$l(\gamma) \geqq \int_\gamma (du^2)^{1/2} = \infty .$$

References

1. T. Klotz, *Surfaces harmonically immersed in* E^3, Pacific J. Math. **21** (1967), 79–87.

2. ———, *Another conformal structure on immersed surfaces of negative curvature*, Pacific J. Math. **13** (1963), 1281–1288.

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