STEENROD OPERATIONS AND TRANSFER

LEONARD EVENS[1]

In this note I show that for the cohomology of a group with coefficients in Z/pZ, the transfer homomorphism commutes with the Steenrod reduced power operations.

Suppose that G is a group and H a subgroup of finite index. Let P denote the cyclic group of prime order p considered as a subgroup of S_p, the symmetric group on p symbols. As in [2], we denote by $S \wr G$, the semi direct product of the permutation group S with the group G^p where the former acts on the latter by permuting the factors. Then $P \times G$, for example, may be considered a subgroup of $P \wr G$ by imbedding G in G^p via the p-fold diagonal map. If this is done, the basic Steenrod construction may be described simply: Given $\alpha \in H^q(G, Z/pZ)$, form the element $1 \wr \alpha \in H^{pq}(P \wr G, Z/pZ)$ as follows. Let X be a G-projective resolution of Z and suppose $f \in \mathrm{Hom}_G(X_q, Z/pZ)$ represents α. Let W be a P-projective resolution of Z with augmentation ϵ. Then $W \otimes X^p$ becomes a $P \wr G$ projective resolution of Z and $\epsilon \otimes f^p$ is a $P \wr G$ homomorphism of $W \otimes X^p$ into $Z \otimes (Z/p)^p \cong Z/pZ$ which is in fact a cocycle whose cohomology class—denoted by $1 \wr \alpha$—depends only on α. (More generally, we may replace P by any subgroup S of S_p, but then it is necessary to include a sign in the action of S on $Z \otimes (Z/pZ)^p$.) Denote by $P(\alpha) = P_G(\alpha) \in H^{pq}(P \times G, Z/pZ)$ the restriction of $1 \wr \alpha$ to the subgroup $P \times G$. $P(\alpha)$ is the basic object from which the reduced powers are constructed.

Suppose next that $\beta \in H^q(H, Z/pZ)$. Let $T = \{\tau\}$ be a left transversal of H in G. Since H is a subgroup of G, X is also an H-projective resolution of Z and if $g \in \mathrm{Hom}_H(X_q, Z/pZ)$ represents β, then $\sum_{\tau \in T} \tau g$ represents $\mathrm{tr}_{H \to G}(\beta)$. (See [1, Chapter XII, §8].) We wish to study $P(\mathrm{tr}\, \beta) = \mathrm{res}(1 \wr (\mathrm{tr}\, \beta))$. As above, $1 \wr \mathrm{tr}\, \beta$ is represented by

$$\epsilon \otimes \left(\sum \tau g\right)^p = \sum \epsilon \otimes \tau_1 g \otimes \tau_2 g \otimes \cdots \otimes \tau_p g \quad (\tau_1, \tau_2, \cdots, \tau_p) \in T^p.$$

We decompose this sum into a sum of terms each of which is a $P \times G$ cocycle. To accomplish this end, let $P \times G$ act on T^p, the first factor by permutation, the second diagonally. Under this action T^p decom-

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poses into a set of disjoint orbits $\{0\}$. Also if a given orbit 0 contains $(\tau_1, \tau_2, \cdots, \tau_p)$ and L is the subgroup of $P\times G$ *fixing* $(\tau_1, \tau_2, \cdots, \tau_p)$, then the contribution to the total sum from this orbit 0 represents the transfer from L to $P\times G$ of some element of $H^*(L, Z/pZ)$.

Consider $P\cap L$. Suppose first that $P\cap L=(1)$. Then the product $PL=M$ is *direct*. We claim that the transfer from L to $P\times L$ is trivial. If this contention is granted, it follows by transitivity that the transfer from L to $P\times G$ is trivial, and the contribution from the given orbit is trivial. On the other hand, the claim itself is valid since everything in $H^*(L, Z/p)$ is the restriction of something in $H^*(P\times L, Z/p)$ and restriction followed by transfer is multiplication by the index which in this case is p.

Suppose instead, that $P\cap L=P$. However, the only elements of T^p which are fixed by P are those of the form $(\tau, \tau, \tau, \cdots, \tau)$ with $\tau\in T$, and the set of all these forms one orbit for $P\times G$. Moreover, in this case, taking $(\tau, \tau, \cdots, \tau)=(1, 1, \cdots, 1)$, we have $L=P\times H$. Hence, the only (possibly) nontrivial contribution to the sum is the transfer from $P\times H$ to $P\times G$ of the element of $H^*(P\times H, Z/pZ)$ represented by $\epsilon\otimes g^p$, that is, the transfer of $P_H(\beta)=\text{res}(1 \wr \beta)$.

Summarizing the above result in a formula, we have

PROPOSITION 1. $\text{tr}_{P\times H\to P\times G}(P_H(\beta))=P_G(\text{tr}_{H\to G}(\beta))$.

To derive the desired result for reduced powers, we note that we can write $P_G(\alpha)=\sum \mu_i\times D^i_G(\alpha)$ where μ_i is an appropriate generator of $H^i(P, Z/p)$ and $D^i_G(\alpha)$ denotes the ith reduced power. (See [3, Chapter 7, §3].) Also $P_H(\beta)=\sum \mu_i\times D^i_H(\beta)$. On the other hand, $\text{tr}_{P\times H\to P\times G}(\mu\times\rho)=\mu\times \text{tr}_{H\to G}(\rho)$ so that we get the desired result.

THEOREM 2. *Let G be a group, H a subgroup of finite index. For each prime p, the Steenrod reduced power operations commute with transfer from H to G.*

REMARK. To gather all the Steenrod operations within the fold, we note that it is a triviality that the Bockstein homomorphism commutes with transfer.

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NORTHWESTERN UNIVERSITY