

NOTE ON THE NONEXISTENCE OF MINIMAL SURFACES

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While a Jordan curve in Euclidean 3-space always bounds a minimal surface of disc-type, it is generally quite difficult to decide whether a frame of several contours is capable of bounding a minimal surface having a prescribed topological type. There is a celebrated criterion, discovered by J. Douglas [4], [5] (see also [3, Chapter IV] and [14], [15]), which guarantees the existence of such a minimal surface in certain instances. But this criterion is only a sufficient one; and there are classical examples—the situation of the catenoid being a case in point (see [2, pp. 85–121] and [8])—which show that minimal surfaces may exist, even though Douglas's condition is violated.

We are concerned here with the special question under which circumstances two distinct Jordan curves Γ_1 and Γ_2 can bound a minimal surface S of the type of the circular annulus. It has recently been shown that the curves Γ_1 and Γ_2 lose this ability if they are too far apart, and quantitative estimates have been given;² see [1], [8], [9], [10], and [11, pp. 238–241]. If Γ_1 and Γ_2 are sufficiently far apart, they can be separated by a slab bounded by parallel planes. Since each curve is contained in a certain sphere, and since a minimal surface is contained in the convex hull of its boundary (see [12, p. 34]), the intersection of S with the boundary planes of the slab are point sets whose diameter can be estimated in terms of the diameters, or the lengths (if these are finite), of Γ_1 and Γ_2 . It is therefore understandable that a discussion of the case where Γ_1 and Γ_2 are curves in parallel planes is of basic importance. It is the purpose of this note to prove the following theorem which, at least by hindsight, is intuitively quite clear:

Let Γ_1 and Γ_2 be two Jordan curves in parallel planes. If there exists a plane which is orthogonal to the planes of these curves and which sepa-

Received by the editors August 10, 1967.

¹ Research sponsored by the Air Force Office of Scientific Research under AFOSR Grant No. 883-67.

² The author announced these results at the Joint Soviet-American Symposium on Partial Differential Equations, Novosibirsk, 1963. While the situation for the catenoid, i.e., the case where the coaxial circles Γ_1 and Γ_2 are moved away from each other vertically, has been discussed in the literature many times, it seems strange that the effect of a lateral shift had never been studied before. (Of course, the first case leads to a one-dimensional problem in the calculus of variations.)

rates them, then Γ_1 and Γ_2 cannot bound a minimal surface of the type of the circular annulus.

Put differently, denoting by Γ_2^1 the orthogonal projection of Γ_2 onto the plane of Γ_1 :

If the (closed) convex hulls of Γ_1 and Γ_2^1 have no common points, then Γ_1 and Γ_2 cannot bound a minimal surface of the type of the circular annulus.

Note that neither the distance of the planes nor the size of Γ_1 and Γ_2 enter into this statement. Our theorem is sharp in the sense that examples of minimal surfaces exist for which the convex hulls of Γ_1 and Γ_2^1 have arbitrarily small intersections of interior points. It also leads to quantitative estimates considerably sharper than the ones obtained before. We finally remark that the simple closed character of the boundary parts Γ_1 and Γ_2 of S is not essential. This can be seen from [10].

For the proof let Γ_1 lie in the plane $z=c_1$ and Γ_2 in the plane $z=c_2 > c_1$. Under the assumptions of the theorem, there exist two discs $D_1 = \{x, y, z; (x-x_1)^2 + (y-y_1)^2 < r_1^2, z=c_1\}$ and $D_2 = \{x, y, z; (x-x_2)^2 + (y-y_2)^2 < r_2^2, z=c_2\}$ containing Γ_1 and Γ_2 , respectively, such that $(x_2-x_1)^2 + (y_2-y_1)^2 > (r_1+r_2)^2$. We shall prove the impossibility of the latter inequality. By the inclusion theorem of [10], if Γ_1 and Γ_2 bound an annulus-type minimal surface S , then the boundaries of the discs D_1 and D_2 , i.e., the circles ∂D_1 and ∂D_2 , must also bound such a surface S_0 . It has been shown by M. Shiffman [16] that this minimal surface S_0 is intersected in circles by all planes $z=c$ ($c_1 < c < c_2$). As it happens all minimal surfaces generated by a one-parameter family of circles have already been determined in the last century by A. Enneper [6] and B. Riemann [13]. A picture can be found in [7, p. 294]. With the exception of the catenoid these surfaces have a representation in terms of elliptic integrals of the first and second kinds; see [6], [7], [13]. After a suitable choice of the constants, and in a suitable coordinate system, it is

$$x = \int_b^u \frac{t^2 dt}{\Delta(t)} + u \cos v, \quad y = u \sin v, \quad z = ab \int_b^u \frac{dt}{\Delta(t)}.$$

Here $0 < b \leq a$ and $\Delta(t) = [(t^2 + a^2)(t^2 - b^2)]^{1/2}$. Obviously, b is the radius of the smallest circle, or gorge, on S_0 which here is contained in the plane $z=0$. (The integrals are two-valued, just as $\cos h^{-1}u$ is.) The curve

$$x = \int_b^u \frac{t^2 dt}{\Delta(t)}, \quad y = 0, \quad z = ab \int_b^u \frac{dt}{\Delta(t)}$$

denotes the locus of the centers of the generating circles whose radius is equal to u . As the z -coordinate of this curve varies from $-z_0$ to z_0 , where $z_0 = (ab/(a^2+b^2)^{1/2})K(a/(a^2+b^2)^{1/2})$, the variable u decreases monotonously from ∞ to b and then increases again monotonously from b to ∞ ; see [7].

We now have

$$\int_b^u \frac{t^2 dt}{\Delta(t)} < \int_b^u \frac{t dt}{(t^2 - b^2)^{1/2}} = (u^2 - b^2)^{1/2} < u.$$

This inequality has the following interpretation: The lateral distance of the centers of any two of the generating circles is less than the sum of their radii. But this stands in contradiction to the inequality $(x_2 - x_1)^2 + (y_2 - y_1)^2 > (r_1 + r_2)^2$ from above. The theorem follows from this fact.

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