EXTENDING HOMEOMORPHISMS BY MEANS OF COLLARINGS

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1. **Introduction.** Let I^∞ denote the Hilbert cube $\prod_{i=1}^{\infty} I_i$, where $I_i = I = [0, 1]$ with metric $\rho(x, y) = \sum_{i=1}^{\infty} |x_i - y_i| / 2^i$. If A, B are subsets of a space X such that $A \subset B$, A is said to be *collared* (*bi-collared*) in B if there is a homeomorphism h carrying $A \times [0, 1)(A \times (0, 1))$ into an open subset of B containing A such that $h(a, 0) = a(h(a, \frac{1}{2}) = a)$ for all $a \in A$. We let $H(X)$ denote the group of all homeomorphisms of X onto itself. It is shown in [**2**] by Wong that if K is a collared sub-Hilbert cube in I^∞, then each homeomorphism h of K onto K_0 can be extended to an $\tilde{h} \in H(I^\infty)$, where $K_0 = \{x \in I^\infty : x_1 = 0\}$. This result also follows from a Theorem in [**1**]. The following question was raised in [**2**]: If K is a bi-collared sub-Hilbert cube in I^∞, is it true that each homeomorphism h of K onto $K_{1/2}$ can be extended to an $\tilde{h} \in H(I^\infty)$, where $K_{1/2} = \{x \in I^\infty : x_1 = \frac{1}{2}\}$? It is the purpose of this paper to answer this question affirmatively. In fact more general situations are studied. We show that if K is a closed subset of I^∞ such that each component of $I^\infty - K$ is of type (Q) (see below for definition), then a necessary and sufficient condition that a homeomorphism h of K into I^∞ be extended to an $\tilde{h} \in H(I^\infty)$ is that h be extended to a neighborhood of K onto a neighborhood of $h(K)$. Theorem 1 gives a necessary and sufficient condition that a closed subset K of I^∞ be mapped onto a point by a mapping of I^∞ onto itself supported on a neighborhood of K and is 1-1 outside of K.

2. **Definition.** An open subset V of I^∞ is of *type* (Q) if the boundary of V, $\mathrm{Bd}(V)$, is collared in $\overline{V}$ and is homeomorphic to I^∞. A mapping is a continuous function. If f is a mapping from a space X into a space Y, K is an *inverse set* of f if K contains at least two points and for some $y \in Y$, $K = f^{-1}(y)$. A mapping f of X into itself is to be *supported* on C if f is the identity on $X - C$. We let e denote the identity mapping for the corresponding space. A closed subset K of I^∞ is of *type* (P) if for each open set U containing K, there is a mapping f of I^∞ onto itself such that f is supported on U, 1-1 outside of K and maps K onto a

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point. A continuum C is *unicoherent* provided that if $C=H\cup K$, H and K are subcontinua, then $H\cap K$ is connected. Throughout this paper, we shall denote the set $\{x\in I^\infty: x_1=t\}$ by K_t.

3. The following lemmas will be needed to prove Theorem 1.

LEMMA 1. *I^∞ is an unicoherent continuum.*

PROOF. Well known. (See for example, [3, p. 364].)

LEMMA 2. *Let K be a bi-collared subcontinuum of I^∞ and let $h(K\times(0,1))$ be a bi-collared of K in I^∞. Then both K and $h(K\times(\frac{1}{3},\frac{2}{3}))$ separate I^∞ into exactly two components.*

PROOF. Let $D_1=h(K\times(0,1/3])$ and $D_2=h(K\times[2/3,1))$. Let $A=h(K\times[1/3,2/3])$ and $B=I^\infty-\text{Int}\,(A)$. Clearly A is connected and $A\cap B$ is not. Therefore by Lemma 1, B must be disconnected. B cannot contain more than two components since each component of B must contain either D_1 or D_2. Hence B has exactly two components. It clearly follows that $I^\infty-K$ also has exactly two components.

LEMMA 3. *Let $V=(t,1]\times\prod_{i=2}^\infty I_i$, where $0<t<1$ and $(t,1]\subset I_1$. Then there is a mapping h of I^∞ onto itself such that h is the identity on K_1, 1-1 on V and maps $I^\infty-V$ onto* 0 $(=(0,0,\cdots))$.

PROOF. Let $Q_0=I^\infty$. For $n\geqq 1$, let $Q_n=\prod_{i=1}^n[0,1/2^n]_i\times\prod_{j=n+1}^\infty I_j$ where each $[0,1/2^n]_i=[0,1/2^n]$ and for $n\geqq 1$, let $C_n=\text{Bd}(Q_n)$. It is evident that (1) $\lim_{n\to\infty}\text{diam}(Q_n)=0$ and (2) for $n>m\geqq 1$, there is a homeomorphism of I^∞ onto itself carrying C_m onto C_n and is the identity on a neighborhood of K_1. Now let $\{t_i\}_{i\geq 0}$ be a sequence of reals such that $t<\cdots<t_2<t_1<t_0=1$ and $t_i\to t$. Evidently for each $i\geqq 1$, we can choose inductively a homeomorphism h_i of $[t_i,t_{i-1}]\times\prod_{j=2}^\infty I_j$ onto $Q_{i-1}-\text{Int}(Q_i)$ so that h_1 is the identity on K_1 and h_i agrees with h_{i+1} on $t_i\times\prod_{j=2}^\infty I_j$. Then define h by taking $h=h_i$ on each $[t_i,t_{i-1}]\times\prod_{j=2}^\infty I_j$ and let $h(I^\infty-V)=0$.

COROLLARY 1. *For each closed subset K of I^∞ and each $x\in I^\infty-K$, there is a neighborhood V of x such that $\overline{V}\cap K=\emptyset$ and for each $\epsilon>0$, there exists an $h\in H(I^\infty)$ satisfying* (1) *h is the identity on V and* (2) diam $(h(K))<\epsilon$.

PROOF. We may assume $x=0$ since I^∞ is homogeneous. Now the corollary clearly follows from the method in proving Lemma 3.

NOTE. We shall from now on denote such a neighborhood V as $N(x,K)$.

COROLLARY 2. *Let $V_0'=\{x\in I^\infty: x_1\in(3/8,5/8)\}$. Then there is a* *mapping f of I^∞ onto itself such that f is the identity on $K_{1/2}$,* 1-1 *on V_0' and map A_0' to* 0, *A_1' to* 1 $(=(1, 1, \cdots))$, *where A_0', A_1' are components of $I^\infty - V_0'$ such that $0 \in A_0'$.*

Proof. Clear.

4. **Theorem** 1. *Let K be a closed subset of I^∞. The following are equivalent*:

(i) *There is $N(x, K)$ (see* §3, *Corollary* 1) *such that for each open set U containing K, there is an open set V such that $K \subset V \subset U$, and an $h \in H(I^\infty)$ such that h is identity on K and carries $I^\infty - V$ into $N(x, K)$.*

(ii) *K is of type (P).*

(iii) *For each open set U containing K, there is an open set V of type (Q) such that $K \subset V \subset U$.*

Proof. (i)⇒(ii). Let U be an open set containing K. Choose a sequence of open neighborhoods $\{U_i\}_{i=1}^\infty$ about K with the following properties: (1) Each $U_i \subset U \cap (I^\infty - \overline{N}(x, K))$, (2) $U_i \supset \overline{U}_{i+1}$, (3) $K = \bigcap_{i=1}^\infty U_i$ and (4) for each i, there is an $h_i \in H(I^\infty)$ carrying $I^\infty - U_i$ into $N(x, K)$ and is the identity on K. We now define a sequence $\{f_i\}_{i=1}^\infty \subset H(I^\infty)$ as follows: For $\epsilon > 0$, let $g_1 \in H(I^\infty)$ such that $g_1|_{N(x,K)} = e$ and $\operatorname{diam}(g_1(K)) < \epsilon$. Let $f_1 = h_1^{-1} g_1 h_1$. Then clearly $f_1|_{I^\infty - U_1} = e$ and we can choose ϵ small enough so that $\operatorname{diam}(f_1(K)) < \frac{1}{2}$. Now suppose f_i has been defined. For $\epsilon > 0$, let $g_{i+1} \in H(I^\infty)$ such that $g_{i+1}|_{N(x,K)} = e$ and $\operatorname{diam}(g_{i+1}(K)) < \epsilon$. Let $f_{i+1} = f_i h_{i+1}^{-1} g_{i+1} h_{i+1} f_i^{-1}$. It is routine to verify that f_{i+1} is the identity on $f_i(I^\infty - U_{i+1})$. Since ϵ is arbitrary, we can choose ϵ small enough so that $f_{i+1}(f_i(K)) < 1/2^{i+1}$. Now let $f = \cdots f_2 f_1$. It is evident that f exists and is the desired mapping.

(ii)⇒(iii). Clear.

(iii)⇒(i). Let $a \in I^\infty - K$. Let U be an open set containing K. Let V be an open set of type (Q) such that $K \subset V \subset U$. We may suppose $\operatorname{cl}(N(a, k)) \cap \overline{V} = \varnothing$. Let $B = \operatorname{Bd}(V)$. Let h be a homeomorphism of $B \times [0, 1)$ onto an open subset of $\overline{V}$ such that $h(b, 0) = b$ for all $b = B$. Let $B_{1/2} = h(B \times \frac{1}{2})$. Since $B_{1/2}$ is homeomorphic to I^∞, there is a homeomorphism g_0 of $h(B \times (0, 1))$ onto V' carrying $B_{1/2}$ onto $K_{1/2}$ where $V' = \{x \in I^\infty : x_1 \in (1/3, 2/3)\}$. Let V_0', A_0', A_1' and f be defined as in §3, Corollary 2. Let $V_0 = g_0^{-1}(V_0')$. Let A_0, A_1 be components of $I^\infty - V_0$ such that $a \in A_0$. We may assume g_0 is so chosen that $g_0(A_0) \cap h(B \times (0, 1)) = A_0' \cap V'$. Define a mapping g of I^∞ onto itself as follows: $g(x) = f g_0(x)$ for $x \in V_0$ and let $g(A_0) = 0$, $g(A_1) = 1$. Then g has exactly two inverse sets A_0 and A_1.

We now proceed to eliminate the inverse set A_1 from g. Let $L' = \operatorname{Bd}([0, 1/8] \times [0, 1/8]) \times \prod_{i=3}^\infty I_i \subset I^\infty$. Let C be the closure of the component of $I^\infty - L'$ containing 1. Let $x' = (1/8, 0, 0, \cdots) \in I^\infty$ and $x = g^{-1}(x')$. Using homogeneity of I^∞ and method of Lemma 3 again, there are a neighborhood W about x and an $\alpha \in H(I^\infty)$ such that (1) $\overline{W} \subset V_0$, (2) $\alpha|_W = e$ and (3) $\alpha(A_0) \subset N(a, K)$. Since g is a local homeomorphism at x, there is an open set W' about x' such that $g^{-1}(W') \subset W$. Evidently there is a homeomorphism β of I^∞ onto C satisying (1) β is the identity on a neighborhood of 1 and (2) $\beta^{-1}(L') \subset W'$. Define

$$F(x) = x \qquad \text{if } x \in A_1,$$
$$= g^{-1}\beta g(x) \qquad \text{if } x \notin A_1.$$

Then (1) F is a mapping of I^∞ into itself, (2) A_0 is the only inverse set of F, (3) if $L = g^{-1}\beta^{-1}(L')$, then $L \subset W$ and $F(L) = g^{-1}(L')$, and (4) $F(I^\infty) = g^{-1}(C) =$ closure of the component of $I^\infty - F(L)$ that does not contain a.

$F(I^\infty - A_1)$ is an open set about the point $F(A_0)$. Let $\gamma \in H(I^\infty)$ such that γ is the identity on a neighborhood of $F(A_0)$ and carries $F(I^\infty)$ into $F(I^\infty - A_1)$. Now define G as follows:

$$G(x) = F^{-1}\gamma F(x) \qquad \text{if } x \notin A_0,$$
$$= x \qquad \text{if } x \in A_0.$$

Then (1) G is a homeomorphism of I^∞ into $I^\infty - A_1$, (2) $G(L) = Bd(G(I^\infty))$ and (3) G is the identity on a neighborhood of A_0. Let $h_0 = G\alpha G^{-1}$. h_0 is a homeomorphism of $G(I^\infty)$ onto itself which is the identity on $\mathrm{Bd}(B(I^\infty))$ and maps A_0 into $N(a,K)$. We can extend h_0 to an $\tilde{h}_0 \in H(I^\infty)$ by letting $\tilde{h}_0$ to be the identity mapping on $I^\infty - G(I^\infty)$. By observing $\tilde{h}_0(I^\infty - V) \subset \tilde{h}_0(A_0) \subset N(a, K)$, we conclude that the proof is completed.

THEOREM 2. *If $V \subset I^\infty$ is an open set of type (Q), then $\overline{V}$ is homeomorphic to I^∞.*

PROOF. Let B, h and $B_{1/2}$ be defined as in proof (iii)⇒(i). There is a homeomorphism f of $h(B \times [0, 1/2])$ onto $V_{1/2}$ carrying B onto K_0, $B_{1/2}$ onto $K_{1/2}$, where $V_{1/2} = \{x \in I^\infty : x_1 \in [0, 1/2]\}$. Let U be the component of $I^\infty - B_{1/2}$ such that $B \cap U = \varnothing$. $\overline{U}$ clearly satisfies (iii), hence is of type (P). Let g be a mapping of I^∞ onto itself such that g is supported on V, 1-1 outside of $\overline{U}$ and maps $\overline{U}$ onto a point. Let g' be a homeomorphism of I^∞ onto itself such that g' is the identity on K_0, 1-1 on $\mathrm{Int}(V_{1/2})$ and maps $I^\infty - \mathrm{Int}(V_{1/2})$ onto 1. Define a homeomorphism F of $\overline{V}$ onto I^∞ as follows:

$$F(x) = g'fg^{-1}(x) \qquad \text{if } x \neq g(\overline{U}),$$
$$= 1 \qquad \text{if } x = g(\overline{U}).$$

THEOREM 3. *If K is a bi-collared sub-Hilbert cube in I^∞, then each homeomorphism h of K onto $K_{1/2}$ can be extended to a $\tilde{h} \in H(I^\infty)$.*

PROOF. This is a consequence of Theorem 2.

THEOREM 4. *If K is a closed subset of I^∞ such that each component of $I^\infty - K$ is of type (Q), then a homeomorphism h of K into I^∞ can be extended to a $\tilde{h} \in H(I^\infty)$ if and only if h can be extended to an open neighborhood of K onto an open neighborhood of $h(K)$.*

PROOF. Suppose h is extended to a homeomorphism h' from a neighborhood of K onto a neighborhood of $h(K)$. Let $\Phi = \{V: V \text{ is a component of } I^\infty - K\}$. For each $V \in \Phi$, let $B_V = \text{Bd}(V)$ and let f_V be a homeomorphism of $B_V \times [0, 1)$ onto an open subset of $\overline{V}$ such that $f_V(b, 0) = b$ for all $b \in B_V$. There is a $t_V > 0$ such that $f(B_V \times [0, 1/t_V]) \subset$ Domain of h'. Let $K' = K \cup (\bigcup_{V \in \Phi} f(B_V \times [0, 1/2t_V]))$. It is evident that (1) K' is compact, (2) K' is connected, (3) each component of $I^\infty - K'$ has a bi-collared boundary which is homeomorphic to I^∞ and (4) for each $V \in \Phi$, $K' - f(B_V \times 1/2t_V)$ is connected. Now by (2), (3) and (4), components of $I^\infty - K'$ are in a natural way one to one corresponding with the components of $I^\infty - h'(K')$. By method used in Theorem 2, h' can be extended to each component of $I^\infty - K'$ onto the corresponding component of $I^\infty - h'(K')$. This observation suffices to conclude the Theorem.

Question. If K is a bi-collared subcontinuum of I^∞ such that the closure of each component of $I^\infty - K$ is homeomorphic to I^∞, is K necessary homeomorphic to I^∞?

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