THE ODD PRIMARY OBSTRUCTIONS TO FINDING A SECTION IN A V_n BUNDLE ARE ZERO[1]

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1. Let ξ be a stable, orientable vector bundle over an m-dimensionable space X. If n is odd and $t<2n-1$, then $\pi_t(V_n)$ is 2-primary, as a simple application of the Serre $\mathcal{C}$-theory shows. As a result, finding a section of the associated V_n bundle to ξ, for $m<2n-1$, depends only on the 2-primary cohomology of X. The purpose of this note is to prove the same result for n even. Since V_n is a mod p n-sphere through dimension $2n-1$ (when n is even and p is an odd prime), it is clear that the p-primary obstructions do not necessarily lie in zero groups. The precise statement of what we can prove requires us to introduce some notation. This is done in the next section.

The method of proof uses the idea of a *modified Postnikov tower* [2] and some of its more recent formulations.

2. It is possible to construct a resolution of V_n

$$
\begin{array}{ccccccccc}
K(G_0) & & K(G_1) & & & & K(G_s) & & \\
\uparrow & & \uparrow & & & & \uparrow & & \\
V_n & \underset{p^1}{\leftarrow} & V_n^1 & \underset{p^2}{\leftarrow} & \cdots & \underset{p^s}{\leftarrow} & V_n^s & \underset{p^{s+1}}{\longleftarrow} & \cdots
\end{array}
$$

with the following notation and properties. Each map p^s is a fiber projection, with fiber $\Omega K(G_{s-1})$, induced by the vertical arrow on its left. When $G=\{G^t\}$ is a graded group, $K(G)=\prod_t K(G^t, t)$ is a product of Eilenberg-MacLane spaces. The graded groups are as follows,

$$
\begin{aligned}
G_0^t &= H_t(V_n; Z) && t < 2n-1 \\
&= 0 && t \geqq 2n-1 \\
G_s^t &= \sum_{p\neq 2} H_t(V_n^s; Z) \otimes Z_p && t < 2n-1 \qquad (1 \leqq s < s_0) \\
&= 0 && t > 2n-1 \\
G_s^t &= H_t(V_n^s; Z) \otimes Z_2 && t < 2n-1 \qquad (s_0 \leqq s) \\
&= 0 && t \geqq 2n-1
\end{aligned}
$$

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where s_0 is the smallest integer such that $\sum_{t,p} H_t(V_n^{s_0};\, Z)\otimes Z_p = 0$ (the sum is over $t<2n-1$ and p an odd prime). We require of this resolution that the cohomology homomorphisms induced by the fiber projections be trivial for certain choices of coefficient groups and in dimensions less than $2n-1$:

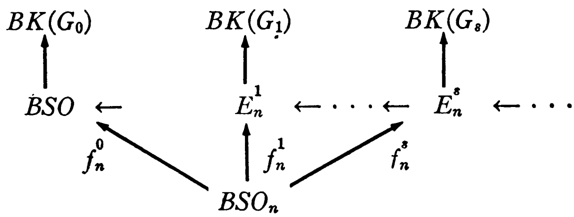
$p^{1*}=0$ with Z for coefficients;

$p^{s*}=0$ with Z_p coefficients, p an odd prime, $s<s_0$;

$p^{s*}=0$ with Z_2 coefficients, for $s\geqq s_0$.

Note that $s_0=1$ when n is odd. When n is even, the existence of s_0 is insured by the fact that $\pi_t(V_n)$ is finitely generated for each t. Indeed, $s_0=n/2$ if n is even. This fact also insures that the resolution has only finitely many terms in it.

Associated with such a resolution is a resolution of BSO_n,



where $BK(G)=\prod_t K(G^t,\, t+1)$. Some details of how this can be done are given in [1].

The primary obstruction to showing that ξ, a bundle over X, has geometric dimension $\leqq n$ (i.e., to lifting ξ to $\xi'\colon X\to BSO_n$) is the obstruction to lifting ξ to $\xi_1\colon X\to E^1$. The *odd primary obstruction* to lifting ξ to ξ' is the obstruction to lifting ξ_1 to $\xi_{s_0}\colon X\to E^{s_0}$. Finally, the 2-*primary obstruction* is the obstruction to lifting ξ_{s_0} to $\xi_\infty\colon X\to E^\infty$. Clearly $BSO_n\to E^\infty$ is a $(2n-2)$-homotopy equivalence. Thus if $\dim X<2n-1$, then the vanishing of these three kinds of obstructions is necessary and sufficient to produce a cross section.

THEOREM. *Let ξ be a stable orientable vector bundle over X, a space of dimension less than $2n-1$, and suppose the primary obstruction to finding a section of the associated V_n bundle vanishes. Then the odd primary obstruction to finding such a section vanishes.*

REMARK. Although we have chosen a particular resolution in order to state our theorem, it is clear that this introduces no loss of generality. Indeed, the theorem shows that any obstruction theory which gives a complete solution to the cross section problem produces a resolution of the homotopy of V_n and, for any such resolution, all obstructions that are not primary but are in a p-primary group are zero.

First observe that if n is odd, the theorem is trivial. From now on we suppose n is even.

LEMMA. *If, in addition, the Stiefel-Whitney class* $w_n(\xi)=0$, *then the conclusion holds.*

PROOF. There exists a natural diagram,

$$\begin{array}{ccccccc} & E_n^1 & \leftarrow \cdots \leftarrow & E_n^{s_0} & \leftarrow \cdots \leftarrow & E_n^{\infty} \\ \swarrow & & & & & \\ BSO & \uparrow & & & & \\ \nwarrow & & & & & \\ & E_{n-1}^1 & & & & \end{array}$$

Note that the space E_{n-1}^1 is the universal example for bundles with $w_n=0$ and vanishing primary obstruction to finding a section of the associated V_n bundle. Thus the Lemma is proved if we show that $E_{n-1}^1\rightarrow E_n^1$ lifts to $E_n^{s_0}$. Suppose we have $\phi\colon E_{n-1}^1\rightarrow E_n^s$ $(s<s_0)$. This gives

$$\begin{array}{ccc} E_{n-1}^1 & \xrightarrow{\phi} & E_n^s \\ f_{n-1}^1 \uparrow & & \uparrow f_n^s \\ BSO_{n-1} & \rightarrow & BSO_n \end{array}$$

Now $\ker (f_{n-1}^1)^*=0 \pmod p$ in dimensions less than $2n-1$. Also note that the k-invariants of $E_n^{s+1}\rightarrow E_n^s$ are all in $\ker (f_n^s)^*$, hence in $\ker \phi^*$, whence ϕ lifts to E_n^{s+1}.

Now we can prove the theorem. Let $\overline{X}\rightarrow X$ be the fibration induced by the map $X\rightarrow K(Z_2, n)$ corresponding to $w_n(\xi)\in H^n(X; Z_2)$. Then the Lemma gives the diagram,

$$\begin{array}{ccc} \overline{X} & \rightarrow & E_n^{s_0} \\ \downarrow & & \downarrow \\ X & \rightarrow & E_n^1 \end{array}$$

We can consider $\overline{X}\subset X$, and the question is then just one of extending a lifting. The obstructions all lie in $H^*(X, \overline{X}; \pi^*)$, where π^* is a group of odd order. Clearly $H^*(X, \overline{X}; \pi^*)=0$, and this completes the proof.

REFERENCES

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