THE GROTHENDIECK GROUP OF FINITELY GENERATED MODULES

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Throughout this paper R will be a commutative Noetherian ring with unit and module will mean an object of $\mathfrak{M} = \mathfrak{M}(R)$, the category of all finitely generated modules over R. We shall denote by $K(R)$ the Euler-Grothendieck group of $\mathfrak{M}$ [**6**, p. 101]. The purpose of this note is to study this group from the point of view of standard ideal theory in R. The first result is an ideal theoretic characterization of $K(R)$. The second, independent of the first, is related to "Grothendieck's Theorem" concerning stability of $K(R)$ under polynomial extensions of R, [**1**], [**2**].

By a *filtration* $\mathfrak{F}$ of a module $E \in \mathfrak{M}$ we mean a finite chain of submodules of E. A *prime-filtration* $\mathfrak{F}$ is one whose factors E_{i+1}/E_i are copies of R/p for various prime ideals p, that is, every E_{i+1}/E_i is cyclic with prime annihilator. It is well known [**3**] that every module has a prime-filtration. Let $\mathfrak{F}$ be a prime-filtration, p a fixed prime ideal. We denote by $n_p(\mathfrak{F})$ the number of times that a factor isomorphic to R/p is associated with $\mathfrak{F}$.

Let $C(R)$ denote the free abelian group generated by the distinct prime ideals p of R, that is the group of all cycles of R [**7**]. The generators will be denoted $\langle p \rangle$. Consider sums of the following form:

$$\sum_p (n_p(\mathfrak{F}) - n_p(\mathfrak{F}'))\langle p \rangle$$

where $\mathfrak{F}$ and $\mathfrak{F}'$ are two arbitrary prime-filtrations of a given module E. Denote by S the subgroup of $C(R)$ generated by all such sums.

LEMMA. $K(R) \cong C(R)/S$.

PROOF. Map $C(R) \to K(R)$ by $\langle p \rangle \to [R/p]$ where $[E]$ represents the element of $K[R]$ corresponding to E. Since every module E has a prime-filtration there exists a filtration $\mathfrak{F}$ of E such that $[E] = \sum_p n_p(\mathfrak{F})[R/p]$. Thus the above map is onto. Further if $\mathfrak{F}$ and $\mathfrak{F}'$ are two prime-filtrations of E we have

$$0 = [E] - [E] = \sum_p (n_p(\mathfrak{F}) - n_p(\mathfrak{F}'))[R/p].$$

Thus S is contained in the kernel. Hence we have a map $C(R)/S \to K(R) \to 0$. Denote cosets mod S by a bar.

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Now we map the category $\mathfrak{M}$ to $C(R)/S$ by $E \to \sum_p n_p(\mathfrak{F})\langle p\rangle$ where $\mathfrak{F}$ is an arbitrary prime-filtration of E. S is constructed so that this is well defined. Suppose $0 \to E' \to E \to E'' \to 0$ is exact. Let $\mathfrak{F}'$, $\mathfrak{F}$, $\mathfrak{F}''$ be arbitrary prime-filtrations of E', E, E'' respectively. Let $\mathfrak{F}_1$ be the prime-filtration of E induced by the two filtrations $\mathfrak{F}'$ and $\mathfrak{F}''$.

Then $\sum_p n_p(\mathfrak{F}')\langle p\rangle + \sum_p n_p(\mathfrak{F}'')\langle p\rangle = \sum_p n_p(\mathfrak{F}_1)\langle p\rangle = \sum_p n_p(\mathfrak{F})\langle p\rangle$. Thus the above map is an Euler-Poincaré map which factors through the universal map $\mathfrak{M} \to K(R)$.

Hence we have a map $K(R) \to C(R)/S$ which is the inverse of $C(R)/S \to K(R) \to 0$ and the lemma is established.

Schreier's theorem [6, p. 103] may be extended to the following

Proposition. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two prime-filtrations of E. Then there are two prime-filtrations $\mathfrak{G}$ and $\mathfrak{G}'$ which are equivalent refinements of $\mathfrak{F}$ and $\mathfrak{F}'$ respectively.*

Proof. By Schreier's theorem we can find two equivalent refinements $\mathfrak{H}$ and $\mathfrak{H}'$ which are ordinary filtrations. Take a fixed prime-filtration of a given factor of $\mathfrak{H}$ and the *same* prime-filtration of the factor of $\mathfrak{H}'$ corresponding to it under the equivalence. These prime-filtrations of the factors of $\mathfrak{H}$ and $\mathfrak{H}'$ induce prime-filtrations $\mathfrak{G}$ and $\mathfrak{G}'$ of E which are equivalent refinements of $\mathfrak{H}$ and $\mathfrak{H}'$ and hence of $\mathfrak{F}$ and $\mathfrak{F}'$ respectively.

Throughout this paper we will be interested in modules of the form $R/(p, x)$ where (p, x) is an ideal generated by a prime ideal p and an element $x \notin p$. We note that $[R/(p, x)] = [R/p] - [(x, p)/p] = [R/p] - [R/p] = 0$ in $K(R)$, since a cyclic submodule of R/p is isomorphic to R/p because p is prime. The following is an important fact about such modules.

Theorem. *Let $\mathfrak{G}$ and $\mathfrak{F}$ be two prime-filtrations of a module E such that $\mathfrak{G}$ is a refinement of $\mathfrak{F}$. Then for all primes p*

$$n_p(\mathfrak{G}) = n_p(\mathfrak{F}) + \sum_{\mathfrak{H} \in \Omega} n_p(\mathfrak{H})$$

where Ω is a finite family of prime-filtrations of modules of the form $R/(q, x)$ for various primes q and elements $x \notin q$. The primes q contain the annihilator of E.

Proof. Let $F_i \in \mathfrak{F}$. Then $F_i = G_j \in \mathfrak{G}$ since $\mathfrak{G}$ is a refinement of $\mathfrak{F}$. F_{i+1}/F_i is isomorphic to R/q for some q containing the annihilator of E. Since $G_{j+1} \subset F_{i+1}$ and $G_{j+1}/G_j = G_{j+1}/F_i$ is cyclic, we have that G_{j+1}/G_j is a cyclic submodule of F_{i+1}/F_i and hence $G_{j+1}/G_j \cong (q, x)/q \cong R/q$ for the same q and some $x \notin q$. Thus $F_{i+1}/G_{j+1} \cong R/(q, x)$. Now

$\mathcal{G}$ induces on F_{i+1}/G_{j+1} a prime-filtration $\mathcal{H}$. Let Ω be the family of such filtrations for every factor of $\mathfrak{F}$. The equation then follows.

Let S be the subgroup defined in the lemma. Since for any q, $x \notin q$, it follows that $[R/(q, x)] = 0 \in K(R)$, we have, for any prime-filtration $\mathcal{H}$ of $R/(q, x)$, that $\sum_p n_p(\mathcal{H})\langle p\rangle \in S$. Conversely we have the following corollary.

COROLLARY 1. *S is generated by elements of the form $\sum_p n_p(\mathcal{H})\langle p\rangle$, where $\mathcal{H}$ is some filtration of $R/(q, x)$ for some q and $x \notin q$.*

PROOF. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two prime-filtrations of a module E. Let $\mathcal{G}$ and $\mathcal{G}'$ be equivalent refinements which are prime-filtrations. Then

$$n_p(\mathfrak{F}) - n_p(\mathfrak{F}') = n_p(\mathcal{G}) - n_p(\mathcal{G}') + \sum_{\mathcal{H}' \in \Omega'} n_p(\mathcal{H}') - \sum_{\mathcal{H} \in \Omega} n_p(\mathcal{H}).$$

Since $n_p(\mathcal{G}) = n_p(\mathcal{G}')$ we have that $\sum (n_p(\mathfrak{F}) - n_p(\mathfrak{F}'))\langle p\rangle$ is a sum of things of the form $\sum_p n_p(\mathcal{H})\langle p\rangle$. Q.E.D.

COROLLARY 2. *For each q, $x \notin q$ choose a particular prime-filtration $\mathfrak{F}$ of $R/(q, x)$. Then S is generated by sums $\sum_p n_p(\mathfrak{F})\langle p\rangle$ quantifying only over q and $x \notin q$.*

PROOF. Suppose not. Let S' be generated by such sums quantifying only over q and $x \notin q$. Let q be maximal among the set of primes such that there exists an $x \notin q$ and a filtration $\mathfrak{F}'$ of $R/(q, x)$ with $\sum_p n_p(\mathfrak{F}')\langle p\rangle$ not in S'. Let $\mathfrak{F}$ be the filtration chosen for this q and x in the statement of the Corollary. Let $\mathcal{G}$ and $\mathcal{G}'$ be equivalent prime refinements of $\mathfrak{F}$ and $\mathfrak{F}'$ respectively. Then

$$n_p(\mathfrak{F}') = n_p(\mathfrak{F}) + n_p(\mathcal{G}') - n_p(\mathcal{G}) + \sum_{\mathcal{H} \in \Omega} n_p(\mathcal{H}) - \sum_{\mathcal{H}' \in \Omega'} n_p(\mathcal{H}'),$$

where each member of Ω and Ω' is a prime-filtration of a module $R/(q', x')$ with q' a prime ideal containing (q, x). Summing over p we have that $\sum_p n_p(\mathfrak{F}')\langle p\rangle$ is a sum of things in S', contradiction.

A few obvious remarks enable us to reduce the set of generators of S still further. Certainly we need quantify only over nonzero cosets mod q instead of elements $x \notin q$. Also we really need quantify only over cosets mod q which are irreducible elements of the domain R/q. This follows from the exact sequence

$$0 \to D/x \xrightarrow{y} D/xy \to D/y \to 0 \qquad \text{for } x, y \neq 0$$

in any domain D. This vaguely suggests why the Grothendieck group is related to factorization questions.

For those familiar with the customary notations in ideal theory we note that finding a prime-filtration of $R/(p, x)$ amounts to finding elements $x_1, \cdots, x_n \in R$ such that $(p, x, x_1, \cdots, x_{i-1}): x_i$ is prime $i=1, \cdots, n$ and such that $(p, x, x_1, \cdots, x_n)=R$.

We give an application of Corollary 2 (see [4]). Let R be a dimension two local U.F.D. with the property that $R/(x, y)$ has even length for every R-sequence x, y contained in the maximal ideal M, and such that there exists one such of length 2. Then $K(R) \cong Z \oplus Z_2$. The generators of $C(R)$ are $\langle (0) \rangle$, $\langle M \rangle$ and various $\langle (p) \rangle$ for principal primes (p). It is easy to see from Corollary 2 that S is generated by things of the form $\langle (p) \rangle$, taking the trivial filtration of $R/(0, p)=R/(p)$; and $n_M(\mathfrak{F})\langle M \rangle$ for filtrations of modules of the form $R/(p, x)$, $p \nmid x$. n_M is always even and at least once 2 by hypothesis. Q.E.D.

We now give a proof of a theorem of R. G. Swan (unpublished) related to Grothendieck's theorem [1], [2]. Proposition 8, of [2], is expressed in terms of varieties and induction on dimension is used. Thus it covers the theorem below only in case R has finite Krull dimension. Swan observed that this assumption could be dispensed with and the proof given here is perhaps the simplest proof of that fact. We note that the group $K(R)$ in this paper is not $K^0(R)$ as defined in [1] unless R is regular. In that paper only the category of projectives is considered. If R is regular of course $K(R)$ is naturally isomorphic to $K^0(R)$.

THEOREM. *Let R be a ring (commutative, noetherian with unit). Then $K(R) \cong K(R[t])$, t an indeterminate.*

PROOF. That the functor $\otimes_R R[t]$ from $\mathfrak{M}(R)$ to $\mathfrak{M}(R[t])$ induces a monomorphism $0 \to K(R) \to K(R[t])$ is known and is in [4]. We will show this map is onto. Under this map $[R/p]$ goes to $[R[t]/p[t]]$. We will show elements of the form $[R[t]/p[t]]$ generate $K(R[t])$. Since we know elements of the form $[R[t]/p']$, p' any prime in $R[t]$, generate, we will express such an element in terms of the others. Let $p' \cap R = p$. Let f be a polynomial of least degree in $p' - p[t]$, a its leading coefficient. a is not in p. By the division algorithm given any $g \in p'$ there exist n and q such that $a^n g = qf + r$. deg $r <$ deg f. It is easy to see $r \in p[t]$.

Consider the ideals $(p[t], f): a^n$ for all n. They form an ascending chain whose union contains p' by the above equation. Since $a \notin p$ the union equals p' and by the a.c.c. there exists an n such that $(p[t], f): a^n = p'$. This tells us that we may construct a prime-filtration of $R[t]/(p[t], f)$ whose bottom factor is $R[t]/p'$ and such that all other factors $R[t]/q$ have the property $(p[t], f, a^n) \subset q$ and thus $a \in q$. Hence $q \cap R$ is a prime ideal of R properly containing p.

Suppose now elements of the form $[R[t]/p[t]]$ generated a proper subgroup G of $K(R[t])$. Let p be maximal among the set of primes of R which are contractions of primes p' in $R[t]$ such that $[R[t]/p']$ is not in G. For this choice of p and p' let f and q be as above. Since $[R[t]/(p[t], f)] = 0$ in $K(R[t])$ it follows that $0 = [R[t]/p] + \sum_q [R[t]/q]$ where $[R[t]/q]$ is in G, contradiction.

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