DENSITY OF INTEGER SEQUENCES

R. KAUFMAN

1. Let \( N = \{ n_k \} \) be an increasing sequence of positive integers. Then \( N \) fulfills condition \( D \) if it contains a sequence of blocks

\[
B_r = [u_r, v_r] \cap N, \quad 1 \leq u_r < v_r,
\]

for which \( v_r - u_r \to \infty \), and \( 1 + v_r - u_r \leq C \cdot |B_r| \). Here \( |S| \) always denotes the number of elements in a finite set \( S \).

Let \( N_1, \ldots, N_r \) be \( r \) sequences fulfilling condition \( D \), and for each real number \( x \), let

\[
A(x) = \{(n_1x, \ldots, n_Rx) : n_1 < n_2 < \cdots < n_r \text{ and } n_s \in N_s \ (1 \leq s \leq r)\}.
\]

Thus \( \Delta(x) \) is a denumerable subset of \( R^r \).

**Theorem 1.** For all but a denumerable set of real numbers \( x \), \( \Delta(x) \) is dense (modulo \( 2\pi \)) in \( R^r \).

The statement for \( r = 1 \) is proved, with more precision, by Amice [1] and Kahane [2]. It will be clear from the proof that the inequalities in the definition of \( \Delta(x) \) can be strengthened almost arbitrarily.

First we express the exceptional set in Theorem 1 as a denumerable union of closed sets. Let \( U_1, \ldots, U_j, \ldots \) be a sequence of open sets in \( R^r \) forming a base for the topology, and \( \Lambda \) the subgroup of \( R^r \) of integral vectors. Put, for each \( j \),

\[
E_j = \{ x \in \mathbb{R} : \Delta(x) \cap (U_j + 2\pi \Lambda) = \emptyset \}.
\]

Then each \( E_j \) is closed and we must show that each is denumerable. In the opposite case some \( E_j \) would contain a compact nondenumerable subset; hence a homeomorph of the Cantor set, and so \( E_j \) would carry a continuous probability measure \( \mu \), \( (\mu(D) = 0 \) for every denumerable set \( D \)). We shall now state a theorem on probability distributions that implies Theorem 1.

2. Let \( M \) be the set of \( r \)-tuples \( (n_1, \ldots, n_r) \in N_1 \times \cdots \times N_r \) defined by the inequalities \( n_1 < n_2 < \cdots < n_r \), and \( X \) a real random variable whose distribution is continuous.

**Theorem 2.** For a certain sequence \( (n_{i1}, \ldots, n_{iw}) \in M \), the \( r \)-dimensional variables \( Y_i = (n_{i1}X, \ldots, n_{iw}X) \), \( i = 1, 2, 3, \ldots \), tend modulo \( 2\pi \) to uniform distribution.

Received by the editors July 25, 1967.
Deduction of Theorem 1. We suppose, after our remarks on Theorem 1, that for some open set $U$ (one of the $U_j$, for example)

$$P\{\Delta(X) \cap (U + 2\pi \Lambda) = \emptyset\} = 1.$$ 

But then $P\{Y_i \subseteq U + 2\pi \Lambda\} = 0$, $i = 1, 2, 3, \ldots$, contradicting the fact that $U + 2\pi \Lambda$ has positive measure in the quotient group $\mathbb{R}/2\pi \Lambda$. This completes the deduction of Theorem 1.

3. Proof of Theorem 2. By Weyl's criterion, a sequence $Y_i$ tends to uniform distribution (modulo $2\pi$) if the $r$-dimensional characteristic functions, $\psi_i$, of the $Y_i$ converge to 0 at each element of $\Lambda$ except 0. For the variables $Y_i$, the characteristic functions $\psi_i$ are determined by the characteristic function $\phi$ of the real variable $X$:

$$\psi_i(m_1, \ldots, m_r) = \phi(m_1n_1 + \cdots + m_rn_r).$$

A theorem of Wiener [3, p. 221] states that

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{j=0}^{m} |\phi(j)| = 0,$$

and in fact the proof cited shows that

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{p}^{p+m} |\phi(j)| = 0,$$

uniformly for all integers $p$.

For any finite set $H$ of elements $\neq 0$ of $\Lambda$, and any $\delta > 0$, we shall find a finite subset $S$ of $M$ so that

$$\sum_{s} \left|\phi(m_1n_1 + \cdots + m_rn_r)\right| < \delta |S|$$

for each $(m_1, \ldots, m_r) \in H$. Then, for at least one $(n_1^*, \ldots, n_r^*) \in S$, and for every $(m_1, \ldots, m_r) \in H$,

$$\left|\phi(m_1n_1^* + \cdots + m_rn_r^*)\right| < \delta |H|$$

this easily implies the existence of the asserted sequence $(n_{i1}, \ldots, n_{ir})$ in $M$. The proof will show that $S$ can be chosen so that (2) holds uniformly for any $(m_1, \ldots, m_r)$ containing at least one coefficient $m_j \neq 0$ but $|m_j| \leq C'$. Given a number $b > 1$, choose, using (1), intervals $[u_1, v_1], \ldots, [u_r, v_r]$ such that

$$b \leq v_1 - u_1, \quad bv_s \leq v_{s+1} - u_{s+1}, \quad 1 \leq s < r,$$

and
1 + v_s - u_s \leq C \cdot |N^s \cap [u_s, v_s]|, \quad 1 \leq s \leq r.

Hence
\[
\prod_{s=1}^{r} (v_s - u_s + 1) \leq C^r \prod_{s=1}^{r} |N^s \cap [u_s, v_s]|.
\]

Let \( Q \) denote the rectangle in \( R^r [u_1, v_1] \times \cdots \times [u_r, v_r] \), so that the last inequality is just
\[
|Q \cap \Lambda| = \prod_{s=1}^{r} (v_s - u_s + 1) \leq C^r |Q \cap N^1 \times \cdots \times N^r|.
\]

Moreover, the inequalities \( n_1 < \cdots < n_r \) are satisfied by all the elements of \( Q \cap \Lambda \) except at most \( rb^{-1}|Q \cap \Lambda| \). Thus if \( b \) is sufficiently large
\[
|Q \cap \Lambda| \leq 2C^r |Q \cap M|.
\]

It is enough to attain, therefore,
(3) \[
\sum_{Q \cap \Lambda} |\phi(m_1n_1 + \cdots + m_rn_r)| \leq \frac{1}{2} \delta C^{-r} |Q \cap \Lambda|.
\]

Suppose, for example, that \( 1 \leq m_r \leq C' \). By holding \( n_1, \cdots, n_{r-1} \) fixed, and varying \( n_r \) in \( [u_r, v_r] \), we obtain from the sums \( m_1n_1 + \cdots + m_rn_r \) an arithmetic progression of at least \( b \) terms, and difference at most \( C' \). So
\[
\sum_{u_r \leq n_r \leq v_r} |\phi(m_1n_1 + \cdots + m_rn_r)| \leq \sup_p \sum_{p}^{p+h} |\phi(j)|,
\]
where \( b \leq v_r - u_{r+1} \leq h \leq C'(v_r - u_r + 1) \). Hence
\[
\sup_p \sum_{p+1}^{p+h} |\phi(j)| = o(h) = o(v_r - u_r + 1) \quad \text{as } b \to \infty.
\]

These estimates are uniform with respect to \( n_1, \cdots, n_{r-1} \) and so (3) holds if \( b \) is sufficiently large. Theorem 2 is completely proved.

References


University of Illinois