A CHARACTERIZATION OF JANKO'S SIMPLE GROUP

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If G is a finite group, we say that a series of subgroups $G=G_0\geqq G_1 \geqq \cdots \geqq G_n=1$ is a maximal series of length n of G if G_i is a maximal subgroup of G_{i-1}, $1\leqq i\leqq n$. A subgroup H of G is called mth maximal in G if there exists at least one maximal series $H=G_m\leqq G_{m-1}\leqq \cdots \leqq G_0=G$. Groups all of whose second, third and fourth maximal subgroups are invariant have been completely classified, see Janko [4], where the relevant results are enumerated. Further, those finite simple groups whose fifth maximal subgroups are trivial have been found by Janko [5].

Since Janko [6] has announced the discovery of a new simple group J, whose sixth maximal subgroups are all trivial, it may be of interest to classify those finite simple groups whose maximal chains are of length at most six. Thompson [8] has given the following

Definition. We say that a finite group G is an N-group if the normalizer of any nontrivial solvable subgroup is itself solvable.

In the Main Theorem of [8], all simple N-groups are classified. These are the following groups: PSL(2, q), q a prime power greater than 3, PSL(3, 3), M_{11}, A_7, $Sz(2^{2n+1})$ and PSU(3, 3^2). Since these groups have been studied elsewhere in great detail, and have been variously characterized group theoretically, we will consider these groups as known. Then we have the

Theorem. *Let G be a finite simple group all of whose sixth maximal subgroups are trivial. Then either G is an N-group or* $G\cong J$.

Remark. It is easy to check that the only simple N-groups all of whose sixth maximal subgroups are trivial are the groups PSL(2, q) for certain prime powers q or the group A_7.

Proof of the Theorem. If G is not an N-group, then G contains a solvable subgroup $S\neq 1$ whose normalizer $N=N(S)$ is nonsolvable. All maximal chains of subgroups of N have length at most 5. Thus all maximal chains of N/S have length at most 4. Since N/S is nonsolvable, Theorems 1 and 2 of Janko [4] show that N/S is simple and $N/S\cong$PSL(2, p), for some prime $p>3$. There is at least one chain of subgroups of length precisely 4 and so S is cyclic of prime order. Also

Received by the editors July 27, 1967.

[1] Supported by NSF grant GP 6132.

we note that p is such that $p-1$ and $p+1$ are products of at most 3 primes and $p\equiv\pm 3 \pmod 8$. Thus an S_2-subgroup of N/S has order 4.

Now since $S \trianglelefteq N(S)$, $C(S) \trianglelefteq N(S)$ and $C(S) \geqq S$. Since N/S is simple, either $C(S)=S$ or $C(S)=N(S)$. However, if $C(S)=S$, then N/S is a subgroup of the automorphism group of S, which is cyclic of order $s-1$. This contradicts the simplicity of N/S. It follows that s divides $p(p^2-1)/2=|N/S|$, for otherwise, S is an S_s-subgroup of G and G is nonsimple by a thoerem of Burnside [3, p. 204].

Let R be an S_2-subgroup of N, $|R| \geqq 4$. We apply the following result of Schur, [9].

Let N be a finite group and let S be a subgroup of N satisfying the following conditions:

(a) $1 \neq S \leqq Z(N)$,

(b) $N/S \cong \mathrm{PSL}(2, p)$, $p>2$, p a prime,

(c) $S \leqq [N, N]$.

Then $|S|=2$ and $N \cong \mathrm{SL}(2, p)$.

Thus we have that either $S \leqq [N, N]$ or $S \cap [N, N]=1$ since S is of prime order. In the first case we have $|S|=2$ and $N \cong \mathrm{SL}(2, p)$. But then an S_2-subgroup R of N is a quaternion group of order 8. Since the centre of R is cyclic, R is also an S_2-subgroup of G. But a group with a quaternion S_2-subgroup is not simple by the theorem of Brauer-Suzuki [1], a contradiction. Thus $S \cap [N,N]=1$, $S \times [N, N] \trianglelefteq N$ and so $N=S \times [N, N] \cong S \times \mathrm{PSL}(2, p)$.

If S is of odd order, $|R|=4$ and R is not an S_2-subgroup of G. For all groups whose S_2-subgroups have order 4 are N-groups by the theorem of Gorenstein and Walter [2]. Consider $M=N(R)$. This group has order divisible by $2^3 \cdot 3 \cdot s$ since $|N(R) \cap N(S)|=2^2 \cdot 3 \cdot s$. Now every fifth maximal subgroup of M is trivial and so every third maximal subgroup of M/R is trivial. By a result of Huppert, [4, Theorem II], M/R is solvable. It follows easily that $|M/R| =2 \cdot 3 \cdot s$, $|M|=2^3 \cdot 3 \cdot s$. Now $|N(R) \cap N(S)|=2^2 \cdot 3 \cdot s$ and so $[N(R)\colon N(R) \cap N(S)]=2$. Thus $N(R) \cap N(S) \trianglelefteq N(R)$. Also $C(R) \cap N(S)=S \times R$ and $C(R) \trianglelefteq N(R)$. Hence $C(R) \cap N(S) \trianglelefteq N(R) \cap N(S)$. Now $S=Z(N(R) \cap N(S))$ and so S is a characteristic subgroup of $N(R) \cap N(S)$. Also R is a normal S_2-subgroup of $N(R) \cap N(S)$ and so is characteristic in $N(R) \cap N(S)$. Thus SR char $N(R) \cap N(S) \trianglelefteq N(R)$. If $s \neq 2$, S char SR and so $S \trianglelefteq N(R)$, $N(R) \leqq N(S)$. This contradicts the fact that an S_2-subgroup of $N(R)$ is of order 8. Hence $|S|=2$ and an S_2-subgroup of N is elementary abelian of order 8.

If R is an S_2-subgroup of G, we may apply the results of Janko [6] and Janko-Thompson [7], to get a contradication if $p \neq 5$ and $G \cong J$

if $p=5$. Thus we may assume that an S_2-subgroup $T\geqq R$ of $N(R)$ is of order at least 16. Since $T\nleqq C(S)$, T is nonabelian.

Since $|N(R)\cap N(S)|=2^3\cdot 3$, $|N(R)|$ is divisible by $2^4\cdot 3$. Now, as before, $N(R)/R$ has the property that every second maximal subgroup is trivial. Hence $N(R)/R$ is solvable and it follows that $|N(R)/R|=2\cdot 3$, $|T|=2^4$. There are just two possible groups T which contain elementary subgroups of order 8.

(a) $T\cong D_8\times C_2$: a direct product of a cyclic group of order 2 and a dihedral group of order 8.

(b) $T\cong\langle a, b, c: a^4=b^2=c^2=1,\ [a, c]=b,\ [a, b]=[b, c]=1\rangle$.

Let X be an S_3-subgroup of $N(R)$. Then if $N(R)$ is 2-closed, X normalizes T. There are just 4 elements of order 4 in the case (a) and 8 elements of order 4 in the case (b). Thus in either case, there exists an element $y\in T$ of order 4 which is fixed by X. But then $[y^2, X]=1$ and since $y^2\in R\leqq T$, $y^2\in C(X)\cap R$. Thus $y^2\in S$ and $y\in C(S)$. This is a contradiction because $C(S)$ has no elements of order 4. Therefore $N(R)/R\cong S_3$.

Now $N(X)\cap N(R)\neq C(X)\cap N(R)$ because otherwise $N(R)$ has a normal 3-complement and is 2-closed. Hence if H is an S_2-subgroup of $N(X)\cap N(R)$, $H\geqq S$, $|H|\geqq 4$. Now $n=[N(R)\colon N(X)\cap N(R)]\equiv 1 \pmod 3$ by Sylow's theorems and so $n=1$ or 4. Of course, $X\ntrianglelefteq N(R)$ because $X\ntrianglelefteq N(R)\cap N(S)$. Therefore $n=4$ and $|H|=4$. It follows that $H\leqq C(S)$ and $C(S)\geqq R, H$. Thus an S_2-subgroup of $C(S)$ has order at least 16, a contradiction. The theorem is proved.

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