THE GEOMETRIC REALIZATION OF A KAN FIBRATION IS A SERRE FIBRATION

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The object of this note is to prove the statement in the title which is asserted without proof in [**1**, Lemma 2.1].

We follow the terminology of [2, II, 3] except that a map of simplicial sets which is both a (Kan) fibration and a weak equivalence will be called an *acyclic fibration* instead of trivial fibration. The term trivial will be used as in [**4**] for a map which is isomorphic to the projection of a product onto one of its factors.

LEMMA. *Any fibration f: $X \to Y$ of simplicial sets may be factored $f = pg$ where p is a minimal fibration and g is an acyclic fibration.*

PROOF. By the theory of minimal fibrations [**3**] (see also [**4**, VI, 5.2]) there is a simplicial subset Z of X such that the restriction p of f to Z is a minimal fibration and such that Z is a strong deformation retract of X relative to Y. Let j be the inclusion of Z in X and let g: $X \to Z$ be the retraction of X onto Z. We claim that g is an acyclic fibration. Suppose given u: $\Delta^{\cdot}[n] \to X$ and v: $\Delta[n] \to Z$ with $gu = vi$, where i is the inclusion $\Delta^{\cdot}[n] \to \Delta[n]$. Recall that the maps in a category form another category with commutative squares for morphisms, and let A (resp. B) be the map from i to f given by the pair u, pv (resp. jgu, pv). The homotopy of deformation from id_X to jg gives a homotopy from A to B. But the map B has a lifting, namely jv, and so by the covering homotopy extension theorem A has a lifting r: $\Delta[n] \to X$. If i_n is the canonical n simplex of $\Delta[n]$, then $gr(i_n)$ and $v(i_n)$ are two simplices of Z which are homotopic relative to their common boundary and to p, so they coincide by the minimality of p. Thus $gr = v$ and $ri = u$ and we find that g is an acyclic fibration. Q.E.D.

The geometric realization of a minimal fibration is a Serre fibration because it is locally trivial [**4**, VI, 5.4] and because the geometric realization of a locally trivial map is a Serre fibration [**4**, VII, 1.4]. As the composition of Serre fibrations is a Serre fibration we are therefore reduced to the case where f is an acyclic fibration.

In this case choose an injective map k: $X \to W$, where W is a contractible Kan complex. This may be done by factoring the map

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$X \to \Delta[0]$ into a cofibration followed by an acyclic fibration. Then $(k, f)\colon X \to W \times Y$ is injective and $\mathrm{pr}_2\colon W \times Y \to Y$ is an acyclic fibration. As $f = \mathrm{pr}_2 \circ (k, f)$ is an acyclic fibration, (k, f) has the left lifting property with respect to f, so f is a retract of pr_2. But $|\mathrm{pr}_2|$ is a Serre fibration because pr_2 is trivial, so $|f|$ is a retract of a Serre fibration and is therefore a Serre fibration. Q.E.D.

REMARK. The above argument may be modified to show that *the geometric realization of a Kan fibration of countable simplicial sets is a Hurewicz fibration.* Indeed one again reduces to proving that if $f\colon X \to Y$ is a locally trivial map of countable simplicial sets with fiber F, then $|f|$ is a Hurewicz fibration. However the proof of [**4**, VI, 5.4] shows that $|Y|$ is covered by open subsets U which are countable CW complexes such that $|f|^{-1}U$ is isomorphic to the product $U \times |F|$ in the category of Kelley spaces. As U and $|F|$ are countable CW complexes, the Kelley product coincides with the ordinary product (use Lemma 2.1 of [**5**]), so $|f|$ is a locally trivial map of spaces. Thus $|f|$ is locally a Hurewicz fibration and since a CW complex is paracompact [**6**], $|f|$ is a Hurewicz fibration [**7**].

We do not know if the countability assumptions are necessary.

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