

THE GEOMETRIC REALIZATION OF A KAN FIBRATION IS A SERRE FIBRATION

DANIEL G. QUILLEN¹

The object of this note is to prove the statement in the title which is asserted without proof in [1, Lemma 2.1].

We follow the terminology of [2, II, 3] except that a map of simplicial sets which is both a (Kan) fibration and a weak equivalence will be called an *acyclic fibration* instead of trivial fibration. The term trivial will be used as in [4] for a map which is isomorphic to the projection of a product onto one of its factors.

LEMMA. *Any fibration $f: X \rightarrow Y$ of simplicial sets may be factored $f = pg$ where p is a minimal fibration and g is an acyclic fibration.*

PROOF. By the theory of minimal fibrations [3] (see also [4, VI, 5.2]) there is a simplicial subset Z of X such that the restriction p of f to Z is a minimal fibration and such that Z is a strong deformation retract of X relative to Y . Let j be the inclusion of Z in X and let $g: X \rightarrow Z$ be the retraction of X onto Z . We claim that g is an acyclic fibration. Suppose given $u: \Delta[n] \rightarrow X$ and $v: \Delta[n] \rightarrow Z$ with $gu = vi$, where i is the inclusion $\Delta[n] \rightarrow \Delta[n]$. Recall that the maps in a category form another category with commutative squares for morphisms, and let A (resp. B) be the map from i to f given by the pair u, pv (resp. jgu, pv). The homotopy of deformation from id_X to fg gives a homotopy from A to B . But the map B has a lifting, namely gv , and so by the covering homotopy extension theorem A has a lifting $r: \Delta[n] \rightarrow X$. If i_n is the canonical n simplex of $\Delta[n]$, then $gr(i_n)$ and $v(i_n)$ are two simplices of Z which are homotopic relative to their common boundary and to p , so they coincide by the minimality of p . Thus $gr = v$ and $ri = u$ and we find that g is an acyclic fibration. Q.E.D.

The geometric realization of a minimal fibration is a Serre fibration because it is locally trivial [4, VI, 5.4] and because the geometric realization of a locally trivial map is a Serre fibration [4, VII, 1.4]. As the composition of Serre fibrations is a Serre fibration we are therefore reduced to the case where f is an acyclic fibration.

In this case choose an injective map $k: X \rightarrow W$, where W is a contractible Kan complex. This may be done by factoring the map

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$X \rightarrow \Delta[0]$ into a cofibration followed by an acyclic fibration. Then $(k, f): X \rightarrow W \times Y$ is injective and $\text{pr}_2: W \times Y \rightarrow Y$ is an acyclic fibration. As $f = \text{pr}_2 \circ (k, f)$ is an acyclic fibration, (k, f) has the left lifting property with respect to f , so f is a retract of pr_2 . But $|\text{pr}_2|$ is a Serre fibration because pr_2 is trivial, so $|f|$ is a retract of a Serre fibration and is therefore a Serre fibration. Q.E.D.

REMARK. The above argument may be modified to show that *the geometric realization of a Kan fibration of countable simplicial sets is a Hurewicz fibration*. Indeed one again reduces to proving that if $f: X \rightarrow Y$ is a locally trivial map of countable simplicial sets with fiber F , then $|f|$ is a Hurewicz fibration. However the proof of [4, VI, 5.4] shows that $|Y|$ is covered by open subsets U which are countable CW complexes such that $|f|^{-1}U$ is isomorphic to the product $U \times |F|$ in the category of Kelley spaces. As U and $|F|$ are countable CW complexes, the Kelley product coincides with the ordinary product (use Lemma 2.1 of [5]), so $|f|$ is a locally trivial map of spaces. Thus $|f|$ is locally a Hurewicz fibration and since a CW complex is paracompact [6], $|f|$ is a Hurewicz fibration [7].

We do not know if the countability assumptions are necessary.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY