A MOTZKIN-TYPE THEOREM FOR CLOSED NONCONVEX SETS

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Introduction. Bouligand [1] recognized the importance of the nearest-points mapping for a closed set X and the set $S_X$ of points with more than one nearest point in $X$ for the study of geometry. Later Motzkin [3], [4] used them in the proof of his theorem characterizing closed convex sets. We use them to show, essentially, that $S_X$ characterizes the complement of $X$ in its convex hull. Our result includes the Motzkin theorem as a special case and yields a theorem of Valentine [5] as a corollary. The original motivation and background for our work can be found in [2].

The statement of the theorem. To every closed set $A$ of the Euclidean $n$-dimensional space $E$ we associate its closed convex hull $C(A)$ and its convex deficiency $D = D(A) = C(A) \setminus A$. We denote by $\pi$ the nearest-points mapping $A$ and by $r$ the distance from $A$:

$$r(x) = d(x, A), \quad \pi(x) = \{y: y \in A, d(x, y) = r(x)\},$$

where $d$ denotes the Euclidean distance.

We let $B(x)$ denote the closed ball around $x$ of radius $r(x)$ and $B^0(x)$ denote its interior. Observe that $B^0(x) \cap A = \emptyset$ and $B(x) \cap A = \pi x$. We shall say that $x \in A$ is a skeletal point of $A$ iff $B(x)$ is contained in no other $B(x')$. The set of all skeletal points of $A$ is the skeleton of $A$ and is denoted by $S$. The skeletal pair of $A$ is $(S, q)$, where $q$ is the restriction of $r$ to $S$. Clearly $S$ contains all points having more than one nearest point in $A$; in fact, as already shown by Motzkin [3], such points form a dense subset of $S$.

Our main result may now be stated.

Theorem. Two closed subsets of $E$ have the same convex deficiency if and only if they have the same skeletal pair.

The proof of the theorem follows.

$D$ determines $(S, q)$. If $x, y \in E$ and $x \neq y$, we let $[x, y]$ denote the segment with endpoints $x$ and $y$ and set $[x, y] = [x, y] \setminus \{y\}$ and $(x, y] = [x, y] \setminus \{x\}$. We let $[y, x)$ denote the closed ray with endpoint
and set \((y, x) = [y, x]\setminus\{y\}\). For \(y \in A\), \(\pi(y) = \{x: x \in E, y \in \pi x\}\). For each set \(X\) we put \(X^* = \{x: x \in E, d(x, C(X)) = d(x, y)\) with \(y \in X\}\). Notice that \(X = C(X) \cap X^*\).

**Lemma 1.** If \(D\) is the convex deficiency of \(A\), we have:

(a) \(D^*\) is the complement of \(A^*\).

(b) \(A^* = A \cup \{x: x \in E, \text{ if } y \neq x \text{ and } y \in \pi x, \text{ then } \pi x = \{y\}\) and \([y, x] \subset \pi^{-1}(y)\}\).

(c) \(D^* = \{x: x \in A, \text{ if } y \in \pi x, \text{ then } \pi^{-1}(y) \cap [y, x] = [y, z]\) for some \(z\}\).

**Proof.** Observe that \(x \in A^*\) iff \(d(x, C(A)) = d(x, A)\). Hence, because \(d(x, C(A)) \leq d(x, A), A^* = E \setminus D^*\) iff \(d(x, C(A)) < d(x, A)\) for each \(x \in D^*\).

If \(d(x, C(A)) < d(x, A)\), then \(d(x, C(A)) = d(x, y)\) for some \(y \in D\). Thus \(d(x, y) \leq d(x, C(A)) \leq d(x, C(D)) \leq d(x, y)\), since also \(y \in C(D)\). Consequently \(x \in D^*\).

Conversely, to prove that \(x \in D^*\) implies \(d(x, C(A)) < d(x, A)\), we prove that \(x \in D^*\) implies \(d(x, C(D)) = d(x, C(A)) = d(x, y)\) for some \(y \in D\). If \(x \in D\), that statement is trivial. Assume then \(x \in D^* \setminus D\), and hence also \(x \in C(D)\). Then, for some \(y \in D\), \(d(x, C(D)) = d(x, y)\). Let \(H_y\) be the hyperplane of support for \(C(D)\) at \(y\) orthogonal to \([y, z]\) and let \(E_y\) be the closed half space bounded by \(H_y\) and containing \(C(D)\). If \(y' \in A \setminus E_y\), then \([y, y'] \subset C(A)\) and, since \(D \subset E_y\), \(y, y' \subset A\). But \(A\) is closed, and hence \(y \in A\), contradicting the fact that \(y \in D\). Then \(A \subset E_y\), \(C(A) \subset E_y\) and \(d(x, C(D)) = d(x, C(A)) = d(x, y)\) with \(y \in D\). If \(d(x, C(A)) = d(x, y) = d(x, A)\), then \(y \in A\) because \(A\) is closed. Hence \(d(x, C(A)) < d(x, A)\) and \((a)\) is established.

To prove \((b)\) it is enough to show that \(A^*\) contains the second set at the right of the equal sign so we pick \(x\) in the set. Then the hyperplane orthogonal to \([y, x]\) passing through \(y\) is a hyperplane of support for \(A\) and hence for \(C(A)\). Thus \(y \in C(A)\) and \(d(x, A) = d(x, C(A))\), that is \(x \in A^*\). Statement \((c)\) follows at once from \((a)\) and \((b)\).

We set \(F(D) = (\text{bd } D) \setminus D\) and observe that \(D = \emptyset\) iff \(D^* = \emptyset\) iff \(F(D) = \emptyset\).

**Lemma 2.** If \(A\) has convex deficiency \(D\), then for \(x \in D^*\) we have \(r(x) = d(x, F(D))\).

**Proof.** Suppose \(y \in \pi x\). If \(x \in D \subset C(A)\), then \((y, x) \subset D, y \in D\), and hence \(y \in F(D)\). If \(x \in D^* \setminus D\), let \(y' = \pi x\) be the projection of \(x\) into \(C(D) \subset C(A)\). Then \([y, y'] \subset C(A)\), \((y, y') \subset C(A) \setminus A = D, y \in D\), and so \(y \in F(D)\).

If \((S, g)\) is the skeletal pair of \(A\), we let \(P(A) = \{\{(y, x): y \in \pi x, x \in S\}\}. We then have the following result:
Lemma 3. Suppose that $A$ has convex deficiency $D$ and skeletal pair $(S, q)$. Then $S \subseteq P(A) = D^\star$.

Proof. The inclusion is trivial. The equality follows from Lemma 1(c) and the observation that $x \in S$ iff $x \in A$ and $\pi^-(y) \cap [y, x] = [y, x]$ for $y \in \pi x$.

The proof of the next lemma is immediate.

Lemma 4. The skeleton $S$ of $A$ is the set of those points $x \in D^\star$ for which

\[ r(x') + d(x, x') = r(x) \quad \text{if} \ x' \in [y, x], \]

\[ r(x) + d(x, x') > r(x') \quad \text{if} \ x' \in [y, x] \setminus [y, x] \]

for every $y \in \pi x$.

We can now establish the first half of the theorem. Let $A$, $A'$ be two closed sets with equal convex deficiency $D$. Then $P(A) = P(A')$ by Lemma 3, and $r(x) = r'(x)$ for each $x \in P(A)$ by Lemma 2. Lemma 4 yields $S = S'$ and consequently $q = q'$.

$(S, q)$ determines $D$. For each set $X$ we put $B^0(X) = \bigcup \{ B^0(x) : x \in X \}$ and $\pi X = \bigcup \{ \pi x : x \in X \}$. Notice that if $X \subseteq A^\circ = \emptyset$, then $B^0(X) \cap A = \emptyset$ and $\pi X \subseteq \text{bd} A$.

Lemma 5. Let $A$ have convex deficiency $D$ and skeletal pair $(S, q)$. Then

(a) $\pi x = B(x) \setminus B^0(S) \subseteq F(D)$ for each $x \in D^\star$.

(b) $\text{Cl} \ \pi S = F(D)$.

Proof. First observe that $D \subseteq B^0(D^\star) \subseteq B^0(S)$. Because $\pi x \subseteq B(x) \setminus B^0(S)$, it is enough to show that $B(x) \setminus B^0(S) \subseteq A$. If $x' \subseteq (B(x) \setminus A) \cap C(A)$, then $x' \subseteq D \subseteq B^0(S)$. Hence $(B(x) \setminus B^0(S)) \cap C(A) \subseteq A$. If $x' \subseteq B(x) \setminus C(A)$, let $y \in \pi x$. Because $y \in C(A)$, $x' \in C(A)$, and $x \in A^\star$, there exists a point $y' \in (y, x') \cap \text{bd} C(A) \subseteq B^0(x)$. Let $y_0 \in \text{bd} C(A) \cap B^0(x)$ be such that $d(x, \text{bd} C^D) = d(x, y_0)$. Clearly, $y_0 \in D$ because $y_0 \in B^0(x)$. Let $H_{y_0}$ be a hyperplane of support for $C(A)$ and hence for $C(D)$ at $y_0$. There exists $z \notin C(D)$ such that the open ray $(y_0, z)$ orthogonal to $H_{y_0}$ at $y_0$ lies in $D^\star \setminus D$. Since, for $z' \subseteq (y_0, z)$, $d(z', y_0) = d(z', C(D)) < d(z', A)$ it is easy to see that $x' \subseteq B^0(z')$ for some $z' \subseteq (y_0, z)$. But $B^0(z') \subseteq B^0(S)$ and hence $(B(x) \setminus B^0(S)) \cap C(A) = \emptyset$ and the equality $\pi x = B(x) \setminus B^0(S)$ is established. By Lemma 2, $\pi x \subseteq F(D)$ and (a) is proved.

From (a) we deduce $\text{Cl} \ \pi S \subseteq F(D)$, since the last set is closed. For the converse, if $y \in F(D)$, then $y \in \text{Cl} D$ and so there exists a sequence $\{ x_n \}$ with $x_n \in D$ and $\{ x_n \} \to y$. Let $y_n \in \pi x_n$; then $d(y, y_n) \leq d(y, x_n)$
+\delta(x_n, y_n). \text{ Since } \{x_n\} \to y, \ r(x_n) \to 0 \text{ and hence also } d(y, y_n) \to 0. \text{ By Lemma 3, } y_n \in \pi S \text{ and thus } y \in \overline{\pi S}.

**Lemma 6.** If \( D \) is the convex deficiency of \( A \), then \( D \subseteq C(F(D)) \) and \( C(D) = C(F(D)) \).

**Proof.** Suppose \( x \in D \) but \( x \notin C(F(D)) \). Project \( x \) onto \( z \in C(F(D)) \) and let \( H_z \) be the supporting hyperplane of \( C(F(D)) \) at \( z \) orthogonal to \([z, x]\) and let \( E_z \) be the corresponding closed half space containing \( C(F(D)) \). We claim that \( C(A) \subseteq E_z \). If not, then there exists some \( y \in A \setminus E_z \). Because \( x \in C(A) \), \([x, y]\) lies in \( C(A) \). Now \([x, y] \cap A\) is closed and nonempty and we let \( y' \) be the unique point of \([x, y] \cap A\) nearest \( x \). Then \([x, y'] \subseteq D\) and \( y' \in F(D) \), contradicting the inclusions \( F(D) \subseteq C(F(D)) \subseteq E_z \). Hence all points of \( A \) and of \( C(A) \) lie in \( E_z \), in particular, \( x \in D \subseteq C(A) \), contradicting our assumption. Hence \( D \subseteq C(F(D)) \). But then \( C(D) \subseteq C(F(D)) \subseteq C(\overline{D}) = C(D) \) and the second statement follows.

We can now terminate the proof of the theorem. Assume that \( A \) and \( A' \) have the same skeletal pair \((S, q)\). Then, by Lemma 5(a), \( \pi x = \pi' x \) for each \( x \in S \) and consequently \( P(A) = P(A') \). Thus by Lemmas 3 and 5(b), \( D^* = D'^* \), \( F(D) = F(D') \). But by Lemma 6 then \( C(D) = C(F(D)) = C(D') \) and \( D = C(D) \cap D^* = D' \).

**References**


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