A MOTZKIN-TYPE THEOREM FOR CLOSED NONCONVEX SETS

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Introduction. Bouligand [1] recognized the importance of the nearest-points mapping for a closed set X and the set S_X of points with more than one nearest point in X for the study of geometry. Later Motzkin [3], [4] used them in the proof of his theorem characterizing closed convex sets. We use them to show, essentially, that S_X characterizes the complement of X in its convex hull. Our result includes the Motzkin theorem as a special case and yields a theorem of Valentine [5] as a corollary. The original motivation and background for our work can be found in [2].

The statement of the theorem. To every closed set A of the Euclidean n-dimensional space E we associate its closed convex hull $C(A)$ and its *convex deficiency* $D=D(A)=C(A)\backslash A$. We denote by π the nearest-points mapping A and by r the distance from A:

$$r(x) = d(x, A), \qquad \pi(x) = \{y: y \in A, d(x, y) = r(x)\},$$

where d denotes the Euclidean distance.

We let $B(x)$ denote the closed ball around x of radius $r(x)$ and $B^0(x)$ denote its interior. Observe that $B^0(x)\cap A=\emptyset$ and $B(x)\cap A=\pi x$. We shall say that $x\notin A$ is a *skeletal point* of A iff $B(x)$ is contained in no other $B(x')$. The set of all skeletal points of A is the *skeleton* of A and is denoted by S. The *skeletal* pair of A is (S, q), where q is the restriction of r to S. Clearly S contains all points having more than one nearest point in A; in fact, as already shown by Motzkin [3], such points form a dense subset of S.

Our main result may now be stated.

THEOREM. *Two closed subsets of E have the same convex deficiency if and only if they have the same skeletal pair.*

The proof of the theorem follows.

D **determines** (S, q). If $x, y\in E$ and $x\neq y$, we let $[x, y]$ denote the segment with endpoints x and y and set $[x, y)=[x, y]\backslash\{y\}$ and $(x, y]=[x, y]\backslash\{x\}$. We let $[y, x\rangle$ denote the closed ray with endpoint

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y and set $(y, x\rangle = [y, x\rangle \backslash \{y\}$. For $y \in A$, $\pi^-(y) = \{x: x \in E, y \in \pi x\}$. For each set X we put $X^* = \{x: x \in E, d(x, C(X)) = d(x, y)$ with $y \in X\}$. Notice that $X = C(X) \cap X^*$.

LEMMA 1. *If D is the convex deficiency of A, we have:*

(a) *D^* is the complement of A^*.*

(b) *$A^* = A \cup \{x: x \in E$, if $y \neq x$ and $y \in \pi x$, then $\pi x = \{y\}$ and $[y, x\rangle \subset \pi^-(y)\}$.*

(c) *$D^* = \{x: x \notin A$, if $y \in \pi x$, then $\pi^-(y) \cap [y, x\rangle = [y, z]$ for some $z\}$.*

PROOF. Observe that $x \in A^*$ iff $d(x, C(A)) = d(x, A)$. Hence, because $d(x, C(A)) \leq d(x, A)$, $A^* = E \backslash D^*$ iff $d(x, C(A)) < d(x, A)$ for each $x \in D^*$.

If $d(x, C(A)) < d(x, A)$, then $d(x, C(A)) = d(x, y)$ for some $y \in C(A) \backslash A = D$. Thus $d(x, y) \leq d(x, C(A)) \leq d(x, C(D)) \leq d(x, y)$, since also $y \in C(D)$. Consequently $x \in D^*$.

Conversely, to prove that $x \in D^*$ implies $d(x, C(A)) < d(x, A)$, we prove that $x \in D^*$ implies $d(x, C(D)) = d(x, C(A)) = d(x, y)$ for some $y \in D$. If $x \in D$, that statement is trivial. Assume then $x \in D^* \backslash D$, and hence also $x \notin C(D)$. Then, for some $y \in D$, $d(x, C(D)) = d(x, y)$. Let H_y be the hyperplane of support for $C(D)$ at y orthogonal to $[y, z]$ and let E_y be the closed half space bounded by H_y and containing $C(D)$. If $y' \in A \backslash E_y$, then $[y, y'] \subset C(A)$ and, since $D \subset E_y$, $(y, y'] \subset A$. But A is closed, and hence $y \in A$, contradicting the fact that $y \in D$. Then $A \subset E_y$, $C(A) \subset E_y$ and $d(x, C(D)) = d(x, C(A)) = d(x, y)$ with $y \in D$. If $d(x, C(A)) = d(x, y) = d(x, A)$, then $y \in A$ because A is closed. Hence $d(x, C(A)) < d(x, A)$ and (a) is established.

To prove (b) it is enough to show that A^* contains the second set at the right of the equal sign so we pick x in the set. Then the hyperplane orthogonal to $[y, x\rangle$ passing through y is a hyperplane of support for A and hence for $C(A)$. Thus $y \in C(A)$ and $d(x, A) = d(x, C(A))$, that is $x \in A^*$. Statement (c) follows at once from (a) and (b).

We set $F(D) = (\text{bd } D) \backslash D$ and observe that $D = \emptyset$ iff $D^* = \emptyset$ iff $F(D) = \emptyset$.

LEMMA 2. *If A has convex deficiency D, then for $x \in D^*$ we have $r(x) = d(x, F(D))$.*

PROOF. Suppose $y \in \pi x$. If $x \in D \subset C(A)$, then $(y, x] \subset D$, $y \notin D$, and hence $y \in F(D)$. If $x \in D^* \backslash D$, let $y' = \pi_0 x$ be the projection of x into $C(D) \subset C(A)$. Then $[y, y'] \subset C(A)$, $(y, y'] \subset C(A) \backslash A = D$, $y \notin D$, and so $y \in F(D)$.

If (S, q) is the skeletal pair of A, we let $P(A) = \bigcup \{(y, x]: y \in \pi x, x \in S\}$. We then have the following result:

LEMMA 3. *Suppose that A has convex deficiency D and skeletal pair (S, q). Then $S \subset P(A) = D^*$.*

PROOF. The inclusion is trivial. The equality follows from Lemma 1(c) and the observation that $x \in S$ iff $x \notin A$ and $\pi^-(y) \cap [y, x\rangle = [y, x]$ for $y \in \pi x$.

The proof of the next lemma is immediate.

LEMMA 4. *The skeleton S of A is the set of those points $x \in D^*$ for which*

$$r(x') + d(x, x') = r(x) \qquad \text{if } x' \in [y, x],$$

$$r(x) + d(x, x') > r(x') \qquad \text{if } x' \in [y, x\rangle \backslash [y, x]$$

for every $y \in \pi x$.

We can now establish the first half of the theorem. Let A, A' be two closed sets with equal convex deficiency D. Then $P(A) = P(A')$ by Lemma 3, and $r(x) = r'(x)$ for each $x \in P(A)$ by Lemma 2. Lemma 4 yields $S = S'$ and consequently $q = q'$.

(S, q) **determines** D. For each set X we put $B^0(X) = \bigcup \{B^0(x) : x \in X\}$ and $\pi X = \bigcup \{\pi x : x \in X\}$. Notice that if $X \cap A = \emptyset$, then $B^0(X) \cap A = \emptyset$ and $\pi X \subset \text{bd } A$.

LEMMA 5. *Let A have convex deficiency D and skeletal pair (S, q). Then*

(a) *$\pi x = B(x) \backslash B^0(S) \subset F(D)$ for each $x \in D^*$.*

(b) *$\text{Cl } \pi S = F(D)$.*

PROOF. First observe that $D \subset B^0(D^*) \subset B^0(S)$. Because $\pi x \subset B(x) \backslash B^0(S)$, it is enough to show that $B(x) \backslash B^0(S) \subset A$. If $x' \in (B(x) \backslash A) \cap C(A)$, then $x' \in D \subset B^0(S)$. Hence $(B(x) \backslash B^0(S)) \cap C(A) \subset A$. If $x' \in B(x) \backslash C(A)$, let $y \in \pi x$. Because $y \in C(A)$, $x' \notin C(A)$, and $x \notin A^*$, there exists a point $y' \in (y, x'\rangle \cap \text{bd } C(A) \subset B^0(x)$. Let $y_0 \in \text{bd } C(A) \cap B^0(x)$ be such that $d(x, \text{bd } C(A)) = d(x, y_0)$. Clearly, $y_0 \in D$ because $y_0 \in B^0(x)$. Let H_{y_0} be a hyperplane of support for $C(A)$ and hence for $C(D)$ at y_0. There exists $z \notin C(D)$ such that the open ray $(y_0, z\rangle$ orthogonal to H_{y_0} at y_0 lies in $D^* \backslash D$. Since, for $z' \in (y_0, z\rangle$, $d(z', y_0) = d(z', C(D)) < d(z', A)$ it is easy to see that $x' \in B^0(z')$ for some $z' \in (y_0, z\rangle$. But $B^0(z') \subset B^0(S)$ and hence $(B(x) \backslash B^0(S)) \backslash C(A) = \emptyset$ and the equality $\pi x = B(x) \backslash B^0(S)$ is established. By Lemma 2, $\pi x \subset F(D)$ and (a) is proved.

From (a) we deduce $\text{Cl } \pi S \subset F(D)$, since the last set is closed. For the converse, if $y \in F(D)$, then $y \in \text{Cl } D$ and so there exists a sequence $\{x_n\}$ with $x_n \in D$ and $\{x_n\} \to y$. Let $y_n \in \pi x_n$; then $d(y, y_n) \leq d(y, x_n)$

$+d(x_n, y_n)$. Since $\{x_n\} \to y$, $r(x_n) \to 0$ and hence also $d(y, y_n) \to 0$. By Lemma 3, $y_n \in \pi S$ and thus $y \in \mathrm{Cl}\ \pi S$.

LEMMA 6. *If D is the convex deficiency of A, then $D \subset C(F(D))$ and $C(D) = C(F(D))$.*

PROOF. Suppose $x \in D$ but $x \notin C(F(D))$. Project x onto $z \in C(F(D))$ and let H_z be the supporting hyperplane of $C(F(D))$ at z orthogonal to $[z, x]$ and let E_z be the corresponding closed half space containing $C(F(D))$. We claim that $C(A) \subset E_z$. If not, then there exists some $y \in A \setminus E_z$. Because $x \in C(A)$, $[x, y]$ lies in $C(A)$. Now $[x, y] \cap A$ is closed and nonempty and we let y' be the unique point of $[x, y] \cap A$ nearest x. Then $[x, y') \subset D$ and $y' \in F(D)$, contradicting the inclusions $F(D) \subset C(F(D)) \subset E_z$. Hence all points of A and of $C(A)$ lie in E_z, in particular, $x \in D \subset C(A)$, contradicting our assumption. Hence $D \subset C(F(D))$. But then $C(D) \subset C(F(D)) \subset C(\mathrm{Cl}\ D) = C(D)$ and the second statement follows.

We can now terminate the proof of the theorem. Assume that A and A' have same skeletal pair (S, q). Then, by Lemma 5(a), $\pi x = \pi' x$ for each $x \in S$ and consequently $P(A) = P(A')$. Thus by Lemmas 3 and 5(b), $D^* = D'^*$, $F(D) = F(D')$. But by Lemma 6 then $C(D) = C(F(D)) = C(D')$ and $D = C(D) \cap D^* = D'$.

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