A SET WHOSE SQUARE CAN MAP ONTO A PERFECT SET

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Assuming the continuum hypothesis, a set X of real numbers will be constructed such that every compact continuous image of X is countable, but $X\times X$ admits a uniformly continuous mapping onto the Cantor set. This completes (modulo the continuum hypothesis and the Ulam measure problem) the determination of whether a product or coproduct of m Boolean algebras or fields of sets can have an infinite free, or projective, or injective sub- or quotient object when the factors do not [1]. The concluding construction (modulo the hypothesis) is embarrassingly easy, but the result is unlike the others [1]; this is the only instance of a finite coproduct creating a remarkable subobject (viz. an infinite free subfield), and there is no instance among these of a finite product creating a remarkable quotient.

The use made of the continuum hypothesis seems slight. However, Sierpiński has pointed out that the weaker result that there is a set of real numbers of the power of the continuum admitting no continuous mapping onto $[0, 1]$ (C_5 in [2]) has not been established without the hypothesis. On the continuum hypothesis, that is a by-product of an important construction of Lusin, the subject of Chapter II of [2].

We shall find X in the Cantor set C considered as an infinite power of the compact group Z_2 (written additively). By a lemma of Lavrentiev (Theorem 99 of [3]), any continuous mapping of a subset of C onto $[0, 1]$ can be extended over a G_δ subset of C. There are only c (the power of the continuum) G_δ sets, and each has only c continuous mappings to $[0, 1]$. Hence we can index all these mappings f_α by ordinals of smaller cardinal than c, and index similarly the points x_α of C. For each f_α, the inverse images of points are c disjoint subsets of C each closed in their union; so except for countably many, they are nowhere dense. By the continuum hypothesis, C is not a union of fewer than c such sets.

There is no difficulty in building up sets S, T, no subset of either of which is mapped onto $[0, 1]$ by any f_α, but with every x_α representable as $s_\alpha+t_\alpha$, s_α in S and t_α in T. Carry along expanding sets H_α disjoint from S, K_α disjoint from T (H_0 and K_0 empty). Arrived

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at α, add to H_α a nowhere dense inverse set of f_α containing none of the fewer than c earlier points s_β, do likewise for K_α, and find s_α in the complement of the union of H_α and the translate $x_\alpha+K_\alpha$. Let $t_\alpha=x_\alpha+s_\alpha$ $(=x_\alpha-s_\alpha)$.

$S\times T$ has the indicated mapping onto C, (s, t) to $s+t$. $S\cup T$ cannot map continuously onto C, since neither S nor T does [1; 3.1]. That proof works as well for $[0, 1]$ as for C; but it seems worth noting that more is true. *A subspace of C admits a continuous map onto C if it admits a continuous map onto* $[0, 1]$. For this, cut out open-closed sets A_i around the inverse image of the ith dyadic rational d_i and modify the map into the ϵ_i-neighborhood of d_i so as to omit an open neighborhood of d_i; if $2\Sigma\epsilon_i<1$, the image must be a Cantor set, and $\epsilon_i\rightarrow 0$ is enough to secure continuity (or uniform continuity, if one began with it). Finally, *a separable metrizable space admits a continuous map onto* $[0, 1]$ *if it has an uncountable compact* (*Hausdorff*) *continuous image*. For it has no uncountable scattered image; a nondegenerate component would map to $[0, 1]$ (which is injective); and a compact, dense-in-itself, totally disconnected space admits a continuous map onto C. Accordingly $X=S\cup T$ is the required example.

The referee observes that the italicized statements above differ little from results in some textbooks. Very likely they have been printed somewhere in at least the present generality. All remarks after the construction of S and T were of course just polishing.

REFERENCES

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