ON RECONSTRUCTING A GRAPH

ROBERT L. HEMMINGER

1. **Introduction.** The term "graph" will here denote an unoriented finite graph without loops or multiple edges. $V(G)$ will denote the vertex set of G and $E(G)$ will denote the edge set. If $a \in V(G)$, we will let G_a denote the graph obtained from G by deleting the vertex a and the edges adjacent to a. If $e \in E(G)$ we will let G^e denote the graph obtained from G by deleting e. P. J. Kelly [3] has proven the following theorem: If G and H are trees and $\sigma: V(G) \rightarrow V(H)$ is a 1-1 onto function such that $G_a \cong H_{\sigma(a)}$ for all $a \in V(G)$, then $G \cong H$. He conjectures that this theorem is true for arbitrary graphs and has verified it for graphs on n vertices where $2 < n \leqq 6$. An equivalent formulation of Kelly's conjecture is as follows: G is uniquely determined, up to isomorphism, by the collection $\{G_a\}_{a \in V(G)}$. We will refer to this as the vertex problem. If a graph G is uniquely determined, up to isomorphism, by a given collection of subgraphs we will say that G can be reconstructed from that collection of subgraphs. It needs to be emphasized that the given subgraphs have no labellings.

Harary and Palmer [1] generalized Kelly's theorem on trees by showing that a tree G can be reconstructed from the G_a with a of degree one in G.

In [2], Harary asks if G can be reconstructed from the collection $\{G^e\}_{e \in E(G)}$. We will refer to this as the edge problem. The purpose of this paper is to show that the edge problem is a special case of the vertex problem.

Undefined terms in the paper can be found in the above-mentioned papers or in [4].

2. **The use of the line graph.** If G is a graph, then the line graph of G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and with $(e_1, e_2) \in E(L(G))$ if and only if e_1 and e_2 are adjacent in G.

LEMMA. *Let G be a given graph. Then $L(G^e) = (L(G))_e$ for all $e \in E(G)$.*

PROOF. Both graphs have $E(G) - \{e\}$ as vertex set, and if e_1, $e_2 \in E(G) - \{e\}$, then the criterion for (e_1, e_2) to be an edge in either graph is the same; namely that e_1 and e_2 be adjacent in G.

Since the number of isolated vertices in G can be discovered from the $\{G^l\}_{e \in E(G)}$ we assume in the following that G and H have no isolated vertices.

Received by the editors October 4, 1967.

THEOREM. *The edge problem is equivalent to the vertex problem for line graphs; i.e., a solution to the edge problem would give a solution to the vertex problem for line graphs and conversely.*

PROOF. Suppose the vertex problem is true for line graphs. Let G and H be graphs and let $\tau: E(G) \to E(H)$ be a 1-1 onto function such that $G^e \cong H^{\tau(e)}$ for all $e \in E(G)$. By the Lemma we then have $(L(G))_e = L(G^e) \cong L(H^{\tau(e)}) = (L(H))_{\tau(e)}$ for all $e \in E(G)$. But then $\tau: V(L(G)) \to V(L(H))$ is a 1-1 onto function such that $(L(G))_e \cong (L(H))_{\tau(e)}$ for all $e \in V(L(G))$ so by our assumption $L(G) \cong L(H)$. Now G and $L(G)$ have the same number of components so G and H have the same number of components and by Whitney's Theorem [5], or see pp. 248 of [4] G and H have the same number of components of each isomorphism type with the possible exception of triangles and 3-pointed stars.

If for each $e \in E(G)$, e is from a triangle component of G if and only if $\tau(e)$ is from a triangle component of H, then $G \cong H$ since they would have the same number of triangle components. If there is some $e \in E(G)$ such that e is from a triangle component but $\tau(e)$ is not then $\tau(e)$ must be from a 3-pointed star component of H. But then $G^e \not\cong H^{\tau(e)}$ since the latter has one more component than the former. (Removing $\tau(e)$ from the star leaves a path of length two and an isolated vertex.) One gets the same contradiction if e is not from a triangle component while $\tau(e)$ is.

The proof that the vertex problem for line graphs is valid if the edge problem is valid is omitted because of its similarity to the above proof.

COROLLARY. *If G is disconnected then G can be reconstructed from the collection $\{G^e\}_{e \in E(G)}$.*

PROOF. $L(G)$ can be constructed from the collection $(L(G))_e$ by [2] since $L(G)$ is disconnected.

It should be pointed out that one can decide from the G^e if G is connected or not. This follows from the observation that G is connected if and only if either G^e is connected for *some* $e \in E(G)$, G^e is a forest with exactly two trees for all $e \in E(G)$ and for some $e \in E(G)$ neither component of G^e is an isolated vertex, or else G^e is a star plus an isolated vertex for each $e \in E(G)$.

REFERENCES

1. F. Harary and E. Palmer, *The reconstruction of a tree from its maximal subtrees*, Canad. J. Math. 18 (1966), 803–811.

2. F. Harary, "On the reconstruction of a graph from a collection of subgraphs," pp. 47–52 in *Symposium on the theory of graphs and its applications* (Prague, 1964), Publ. House. Czech. Acad. Sci., Academic Press, New York, 1964.

3. P. J. Kelly, *A congruence theorem for trees*, Pacific J. Math. **7** (1957), 961–968.

4. Oystein Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., Vol. 38, Amer. Math. Soc., Providence, R.I., 1962.

5. H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150–168.

VANDERBILT UNIVERSITY