1. Introduction. The term “graph” will here denote an unoriented finite graph without loops or multiple edges. \( V(G) \) will denote the vertex set of \( G \) and \( E(G) \) will denote the edge set. If \( a \in V(G) \), we will let \( G_a \) denote the graph obtained from \( G \) by deleting the vertex \( a \) and the edges adjacent to \( a \). If \( e \in E(G) \) we will let \( G^e \) denote the graph obtained from \( G \) by deleting \( e \). P. J. Kelly [3] has proven the following theorem: If \( G \) and \( H \) are trees and \( \sigma : V(G) \to V(H) \) is a 1-1 onto function such that \( G_a \cong H_{\sigma(a)} \) for all \( a \in V(G) \), then \( G \cong H \). He conjectures that this theorem is true for arbitrary graphs and has verified it for graphs on \( n \) vertices where \( 2 < n \leq 6 \). An equivalent formulation of Kelly’s conjecture is as follows: \( G \) is uniquely determined, up to isomorphism, by the collection \( \{ G_a \}_{a \in V(G)} \). We will refer to this as the vertex problem. If a graph \( G \) is uniquely determined, up to isomorphism, by a given collection of subgraphs we will say that \( G \) can be reconstructed from that collection of subgraphs. It needs to be emphasized that the given subgraphs have no labellings.

Harary and Palmer [1] generalized Kelly’s theorem on trees by showing that a tree \( G \) can be reconstructed from the \( G_a \) with \( a \) of degree one in \( G \).

In [2], Harary asks if \( G \) can be reconstructed from the collection \( \{ G^e \}_{e \in E(G)} \). We will refer to this as the edge problem. The purpose of this paper is to show that the edge problem is a special case of the vertex problem.

Undefined terms in the paper can be found in the above-mentioned papers or in [4].

2. The use of the line graph. If \( G \) is a graph, then the line graph of \( G \), denoted by \( L(G) \), is the graph with \( V(L(G)) = E(G) \) and with \( (e_1, e_2) \in E(L(G)) \) if and only if \( e_1 \) and \( e_2 \) are adjacent in \( G \).

**Lemma.** Let \( G \) be a given graph. Then \( L(G^e) = (L(G))^e \) for all \( e \in E(G) \).

**Proof.** Both graphs have \( E(G) - \{ e \} \) as vertex set, and if \( e_1, e_2 \in E(G) - \{ e \} \), then the criterion for \( (e_1, e_2) \) to be an edge in either graph is the same; namely that \( e_1 \) and \( e_2 \) are adjacent in \( G \).

Since the number of isolated vertices in \( G \) can be discovered from the \( \{ G^e \}_{e \in E(G)} \) we assume in the following that \( G \) and \( H \) have no isolated vertices.

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Theorem. The edge problem is equivalent to the vertex problem for line graphs; i.e., a solution to the edge problem would give a solution to the vertex problem for line graphs and conversely.

Proof. Suppose the vertex problem is true for line graphs. Let $G$ and $H$ be graphs and let $\tau: E(G) \to E(H)$ be a 1-1 onto function such that $G^e \cong H^\tau(e)$ for all $e \in E(G)$. By the Lemma we then have $(L(G))_e = L(G^e) \cong L(H^\tau(e)) = (L(H))_{\tau(e)}$ for all $e \in E(G)$. But then $\tau: V(L(G)) \to V(L(H))$ is a 1-1 onto function such that $(L(G))_e \cong (L(H))_{\tau(e)}$ for all $e \in V(L(G))$ so by our assumption $L(G) \cong L(H)$. Now $G$ and $L(G)$ have the same number of components so $G$ and $H$ have the same number of components and by Whitney's Theorem [5], or see pp. 248 of [4] $G$ and $H$ have the same number of components of each isomorphism type with the possible exception of triangles and 3-pointed stars.

If for each $e \in E(G)$, $e$ is from a triangle component of $G$ if and only if $\tau(e)$ is from a triangle component of $H$, then $G \cong H$ since they would have the same number of triangle components. If there is some $e \in E(G)$ such that $e$ is from a triangle component but $\tau(e)$ is not then $\tau(e)$ must be from a 3-pointed star component of $H$. But then $G^e \not\cong H^\tau(e)$ since the latter has one more component than the former. (Removing $\tau(e)$ from the star leaves a path of length two and an isolated vertex.) One gets the same contradiction if $e$ is not from a triangle component while $\tau(e)$ is.

The proof that the vertex problem for line graphs is valid if the edge problem is valid is omitted because of its similarity to the above proof.

Corollary. If $G$ is disconnected then $G$ can be reconstructed from the collection \{ $G^e$ \}_e \in E(G).

Proof. $L(G)$ can be constructed from the collection $(L(G))_e$ by [2] since $L(G)$ is disconnected.

It should be pointed out that one can decide from the $G^e$ if $G$ is connected or not. This follows from the observation that $G$ is connected if and only if either $G^e$ is connected for some $e \in E(G)$, $G^e$ is a forest with exactly two trees for all $e \in E(G)$ and for some $e \in E(G)$ neither component of $G^e$ is an isolated vertex, or else $G^e$ is a star plus an isolated vertex for each $e \in E(G)$.

References


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