TENSOR PRODUCTS OF SIMPLE PURE INSEPARABLE FIELD EXTENSIONS

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Let K be a field of characteristic $p \neq 0$ and let L be a pure inseparable extension field of finite degree over K. Our purpose is to give several necessary and sufficient conditions for L to be a tensor product of simple extensions over K. Weisfeld [4] has a criterion, namely the existence of a nontrivial higher derivation of L with K as its subfield of constants, (in fact Weisfeld proves his criterion for infinite extensions of bounded exponent). The present note describes different criteria, in terms of Pickert's canonical generators [3, p. 133]. For a given canonical generating set $\{b_1, \cdots, b_r\}$ of L over K, let $M_i = K(b_1, \cdots, b_i)$ and let q_i denote p^{e_i} where e_i is the exponent of b_i over M_{i-1}, $i = 1, \cdots, r$, where $M_0 = K$. We shall prove the following theorem.

THEOREM. *If L is a finite degree pure inseparable extension of K, then the following conditions are equivalent*:

(0) *L is the tensor product of a finite number of simple extensions with respect to K.*

(1) *Every canonical generating set is such that*

$$b_i^{q_i} \in (L^{q_i} \cap K)(b_1^{q_i}, \cdots, b_{i-1}^{q_i}) = M_{i-1}^{q_i}(L^{q_i} \cap K), \qquad i = 1, \cdots, r.$$

(2) *Every canonical generating set is such that the tensor product $L \otimes M_i$ with respect to K cleaves over $1 \otimes M_i$ (that is, $L \otimes M_i$ has a Wedderburn factor as an algebra over $1 \otimes M_i$), $i = 1, \cdots, r$.*

(3) *There exists a canonical generating set such that $L \otimes M_i$ cleaves over $1 \otimes M_i$, $i = 1, \cdots, r$.*

(4) *There exists a canonical generating set such that $b_i^{q_i} \in M_{i-1}^{q_i}(L^{q_i} \cap K)$, $i = 1, \cdots, r$.*

PROOF. (0) implies (1): Suppose $L \cong K(a_1) \otimes \cdots \otimes K(a_r)$ and that $\{a_1, \cdots, a_r\}$ is already ordered so that it is a canonical generating set. Let $\{b_1, \cdots, b_r\}$ be any given canonical generating set. For any $c \in L$, $c^{q_i} = (\sum_j k_j a_1^{j_1} \cdots a_r^{j_r})^{q_i}$ where $k_j \in K$ and $j = \{j_1, \cdots, j_r\}$. By the division algorithm, $a_n^{j_n q_i} = a_n^{s_n q_n} a_n^{r_n}$ where $0 \leqq r_n < q_n$ $(n = 1, \cdots, i-1)$. Since q_i divides q_n, r_n has the form $q_i t_n$ $(n = 1, \cdots, i-1)$. Thus, since $\{a_i^{q_i}, \cdots, a_r^{q_i}\} \subseteq K$ and $\{a_1^{q_1}, \cdots, a_{i-1}^{q_{i-1}}\} \subseteq L^{q_i} \cap K$, there

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exists a set $\{k_t'\}\subseteq L^{q_i}\cap K$ such that for $t=\{t_1,\cdots,t_{i-1}\}$,

$$(*)\quad c^{q_i}=\sum_t k_t' a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}},\qquad 0\leqq q_it_n<q_n\ (n=1,\cdots,i-1).$$

Since the monomials $\{a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}}\}$ are linearly independent over K, this set $\{k_j'\}$ is the only subset of K satisfying (*). In particular,

$$(**)\quad b_i^{q_i}=\sum_t k_t a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}},\quad k_t\in L^{q_i}\cap K,\quad t=\{t_1,\cdots,t_{i-1}\}$$

and $0\leqq q_it_n<q_n$ $(n=1,\cdots,i-1)$. Also,

$$b_i^{q_i}=\sum_s k_s'' b_1^{q_is_1}\cdots b_{i-1}^{q_is_{i-1}},\qquad k_s''\in K,\qquad s=\{s_1,\cdots,s_{i-1}\}$$

and

$$0\leqq q_is_n<q_n\qquad (n=1,\cdots,i-1).$$

Thus, by (*),

$$(***)\quad \begin{aligned} b_i^{q_i}&=\sum_s k_s''(b_1^{s_1}\cdots b_{i-1}^{s_{i-1}})^{q_i}\\ &=\sum_s k_s''\left(\sum_t k_{st}a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}}\right),\end{aligned}$$

$k_{st}\in L^{q_i}\cap K$ and $0\leqq q_it_n<q_n$ $(n=1,\cdots,i-1)$. Therefore, by (**) and (***), $k_t=\sum_s k_s''k_{st}$ for each t. Since the set $\{k_s''\}$ exists, the system $k_t=\sum_s x_sk_{st}$ has a solution in $L^{q_i}\cap K$, say $x_s=k_s^*\in L^{q_i}\cap K$. Hence,

$$\begin{aligned} b_i^{q_i}&=\sum_t\left(\sum_s k_s^*k_{st}\right)a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}}\\ &=\sum_s k_s^*\left(\sum_t k_{st}a_1^{q_it_1}\cdots a_{i-1}^{q_it_{i-1}}\right)\\ &=\sum_s k_s^*(b_1^{s_1}\cdots b_{i-1}^{s_{i-1}})^{q_i}\in M_{i-1}^{q_i}(L^{q_i}\cap K).\end{aligned}$$

(4) implies (0): Make the induction hypothesis that $L\cong M_i\otimes M_i'$ where $M_i'=K(a_{i+1})\otimes\cdots\otimes K(a_r)$ (there being nothing to prove for $i=r$). Since $b_i^{q_i}=\sum_j k_jb_1^{q_ij_1}\cdots b_{i-1}^{q_ij_{i-1}}$ where $k_j=c_j^{q_i}\in L^{q_i}\cap K$ and $j=\{j_1,\cdots,j_{i-1}\}$, we have $b_i=\sum_j c_jb_1^{j_1}\cdots b_{i-1}^{j_{i-1}}$. Hence, $M_i=M_{i-1}(b_i)=M_{i-1}(\{c_j\})$. Since M_i is simple pure inseparable over M_{i-1}, there exists $a_i\in\{c_j\}$ such that $M_i=M_{i-1}(a_i)$ and $a_i^{q_i}\in K$. Since $[M_i:K]=q_1\cdots q_i$, it follows that $[M_{i-1}(a_i):M_{i-1}]=q_i$.

Hence, $M_i \cong M_{i-1} \otimes K(a_i)$. Thus, $L \cong M_{i-1} \otimes M'_{i-1}$ where $M'_{i-1} = K(a_i) \otimes \cdots \otimes K(a_r)$. Hence, by induction, $L \cong K(a_1) \otimes \cdots \otimes K(a_r)$. (E. A. Hamann has a different proof of this implication.)

Since (1) implies (4) trivially, we have the equivalence of (0), (1) and (4).

(1) implies (2): Since $b_i^{q_i} \in M_{i-1}^{q_i}(L^{q_i} \cap K)$, $i=1, \cdots, r$, we have $b_i = \sum_j c_j m_j$ where $c_j \in L$, $m_j \in M_{i-1}$ and $c_j^{q_i} = k_j \in K$. Let $b'_i = \sum_j c_j \otimes m_j$. Then $b_i'^{q_i} = \sum_j c_j^{q_i} \otimes m_j^{q_i} = \sum_j 1 \otimes k_j m_j^{q_i} \in 1 \otimes M_{i-1}$. Since e_i is the exponent of b_i over M_{i-1}, $M'_i = (1 \otimes M_{i-1})[b'_i]$ is a field $(i=1, \cdots, r)$.

Now consider $L \otimes M_i$ for any $i=1, \cdots, r$. Suppose there exists a field M_j^* in $L \otimes M_i$ such that $M_j^* \supseteq 1 \otimes M_i$ $(j \geqq i)$ and $f_i M_j^* = M_j$ where f_i is the canonical K-epimorphism of $L \otimes M_i$ onto $LM_i = L$. By the previous paragraph, there exists a field $M'_{j+1} \subseteq L \otimes M_j$ such that $M'_{j+1} \supseteq 1 \otimes M_j$ and $f_j M'_{j+1} = M_{j+1} \subseteq L$. By the universal mapping theorem for tensor products, there exists a K-epimorphism h_j of $L \otimes M_j$ onto the ring composite $[L \otimes 1, M_j^*] \subseteq L \otimes M_i$ such that $f_j = f_i h_j$. Thus, there exists a field M_{j+1}^* in $L \otimes M_i$ such that $M_{j+1}^* \supseteq M_j^* \supseteq 1 \otimes M_i$ and $f_i M_{j+1}^* = M_{j+1} \subseteq L$, namely the field $M_{j+1}^* = h_j M'_{j+1}$. Hence, the proof follows by induction.

(2) implies (3): Immediate.

(3) implies (4): Let $\{b_1, \cdots, b_r\}$ be any canonical generating set such that $L \otimes M_i$ cleaves over $1 \otimes M_i$, $i=1, \cdots, r$. Use the symbol $\otimes_1$ to denote the tensor product with respect to M_1. Then there is a canonical K-epimorphism of $L \otimes M_i$ onto $L \otimes_1 M_i$, whence $L \otimes_1 M_i$ cleaves over $1 \otimes_1 M_i$, $i=1, \cdots, r$. Now make the induction hypothesis that (3) implies (4) for all pure inseparable extensions of multiplicity less than r. ((3) implies (4) trivially for $r=1$.) Then since we have proved (4) is equivalent to (0), $L \cong M_1(b'_2) \otimes_1 \cdots \otimes_1 M_1(b'_{r'})$ and we may assume $\{b'_2, \cdots, b'_{r'}\}$ is canonically ordered over M_1. Since b_1 has maximal exponent in L over K and b'_i has maximal exponent in L over M'_{i-1} ($M'_1 = M_1 = K(b_1)$ and $M'_j = M_1(b'_2, \cdots, b'_j)$, $j=2, \cdots, r'$), it follows that $\{b_1, b'_2, \cdots, b'_{r'}\}$ is a canonical generating set of L over K, whence $r=r'$. In particular,

$$b_j'^{q_j} \in K(b_1^{q_j}, b_2'^{q_j}, \cdots, b_{j-1}'^{q_j}) \cap M_1, \qquad j = 2, \cdots, r,$$

since the e_i of a canonical generating set are invariant. Because $L \otimes M_1$ cleaves over $1 \otimes M_1$, there exists $b_j^* \in L \otimes M_1$ such that $f_1 b_j^* = b'_j$ and $b_j^{*q_j} \in 1 \otimes M_1$, $j=2, \cdots, r$. Now $b_j^* = \sum_s c_s \otimes b_1^s$, $c_s \in L$, whence $b_j^{*q_j} = \sum_s c_s^{q_j} \otimes b_1^{q_j s}$. By the division algorithm, $b_1^{q_j s} = b_1^{q_1 n_s} b_1^{r_s}$ where $0 \leqq r_s < q_1$. Since $b_1^{q_1 n_s} \in K$ and q_j divides q_1, it follows that

$b_j^{*q_j} = \sum_s c_s'^{\,q_j} \otimes b_1^{r_s}$ where $c_s' \in L$. Also, $b_j^{*q_j} = \sum_s 1 \otimes k_s b_1^{q_j s_1} b_2'^{q_j s_2} \cdots b_{j-1}'^{q_j s_{j-1}}$, $s = \{s_1, \cdots, s_{j-1}\}$, since $M_1 \subseteq M_j'$. Therefore, $k_s \in L^{q_j} \cap K$ and $b_j'^{\,q_j} \in M_{j-1}'^{q_j}(L^{q_j} \cap K)$. Hence, $\{b_1, b_2', \cdots, b_r'\}$ is a canonical generating set satisfying (4). Q.E.D.

Examples where L is not a tensor product of simple extensions can be found in [**1**, Ex. 6, p. 196] and [**2**, p. 51]. If $\{b_1, \cdots, b_r\}$ is a canonical generating set satisfying (4), it does not follow that $L \cong K(b_1) \otimes \cdots \otimes K(b_r)$. For example, consider a perfect field P and independent indeterminates s, t over P. Let $K = P(s, t)(s^{1/p} + t^{1/p})$ and $L = P(s^{1/p^2}, t^{1/p^2})$. If $b_1 = s^{1/p^2}$ and $b_2 = t^{1/p^2}$, then $\{b_1, b_2\}$ is a canonical generating set with $e_1 = 2$ and $e_2 = 1$. It is easily verified that $b_2^{q^2} \in (L^{q^2} \cap K)(b^{q^2})$, but $L \not\cong K(b_1) \otimes K(b_2)$ since b_2 has exponent 2 over K. However, $L \cong K(s^{1/p^2}) \otimes K(s^{1/p^2} + t^{1/p^2})$.

The extent to which these results are valid in arbitrary pure inseparable extensions is considered by the authors in an article to appear in the Mathematische Zeitschrift. Other recent results can be found in an article by Haddix and Mordeson in the Formosan Science and in an article by Sweedler in the Annals. The equivalence of (0) above and the linear disjointness of K and L^{p^i} $(i = 1, 2, \cdots)$ is proved independently in these articles.

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