ON QUASI-LOCAL NOETHERIAN RINGS

GERHARD MICHLER[1]

It is the purpose of this note to show that each semiprime, quasi-local, noetherian ring with gl. dim $R \leqq 2$ is Morita equivalent to a quasi-local noetherian domain D with gl. dim $D \leqq 2$ (cf. Theorem 1).

All rings considered here have an identity element; all modules are assumed to be unitary. The ring R is noetherian if R satisfies the ascending chain conditions for right and for left ideals. A domain is a ring without zero-divisors $z \neq 0$. The ring R is quasi-local, if its Jacobson radical J is its unique maximal two-sided ideal.

Our result here is another consequence of the Morita Theorems (cf. Auslander and Goldman [1, Appendix]). In order to apply them we need the following standard notation:

If P is a finitely generated right R-module, and $T = \operatorname{End}_R(P)$, then P is also a left T-module. The map

$$\tau\colon \operatorname{Hom}_R(P, R) \otimes_T P \to R$$

which is defined by $\tau(f \otimes x) = f(x)$ for all $x \in P$ and all $f \in \operatorname{Hom}_R(P, R)$ is called the trace mapping of the R-module P. The image $\tau_R(P)$ of τ is the trace ideal of P. One statement of the Morita Theorems is that $\tau_R(P)$ is an idempotent, two-sided ideal of R, if P is a finitely generated projective right R-module. In case we also have $\tau_R(P) = R$, then P is a finitely generated projective left T-module, and $R \cong \operatorname{End}_{\mathfrak{T}}(P)$.

Theorem 1. *The ring R is a semiprime, quasi-local, noetherian ring with* gl. dim $R \leqq 2$ *if and only if R is isomorphic to the full ring of endomorphisms* $\operatorname{End}_D(P)$ *of a finitely generated projective left D-module P over a quasi-local, noetherian domain with* gl. dim $D \leqq 2$.

Proof. If R is a semiprime noetherian ring, then R has a uniform right annihilator $P \neq 0$ by Goldie [2, p. 205, Theorem 2.3]. Hence $P = t_r = \{x \in R \mid tx = 0\}$ for some $0 \neq t \in R$ by Goldie [2, p. 208, Theorem 3.7]. Since gl. dim $R \leqq 2$, the following standard exact sequence

$$0 \leftarrow R/tR \leftarrow R \;\nwarrow\; tR \;\swarrow\; R \;\nwarrow\; t_r = P \;\swarrow\; 0$$

shows that P is a projective right R-module, which is finitely gen-

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erated. Thus $0 \neq \tau_R(P) = S$ is an idempotent ideal of R. Since R is quasi-local, either $S \leqq J$ or $S = R$ where J is the Jacobson radical of R. Therefore $S = SJ = 0$ by Nakayama's Lemma, in case $S \leqq J$. This implies $S = R$, and so $R \cong \mathrm{End}_D(P)$ by the above remarks, where $D = \mathrm{End}_R(P)$. Furthermore, P is a finitely generated projective left D-module. If the right R-module P is generated by n elements, then P is a direct summand of a free right R-module F on n generators. Let $B = \mathrm{End}_D(F)$. Then there is an idempotent $0 \neq e \in B$ such that $D \cong e\,B\,e$. Clearly, B is quasi-local and noetherian, and gl. dim $B \leqq 2$ by Harada [**3**, Theorem 2]. Hence $B\,e\,B = B$, which implies again by the Morita Theorems that eB is a projective left $e\,B\,e$-module. Thus gl. dim $D =$ gl. dim $(e\,B\,e) \leqq$ gl. dim $B \leqq 2$ by Harada [**3**, p. 27, Theorem 8]. Obviously D is quasi-local and noetherian. Since D is the full ring of R-endomorphisms of the uniform right ideal P of R, D is a domain by Goldie [**2**, p. 218, Theorem 5.6]. This completes the proof of Theorem 1, because the converse part is now obvious.

COROLLARY 1. *A semiprime, quasi-local, noetherian ring R with* gl. dim $R \leqq 2$ *is a prime ring.*

The *proof* follows at once from Theorem 1, because it states that R is Morita equivalent to a domain.

REMARK. We do not know whether Theorem 1 holds, if we drop the requirement that R be semiprime, but assume that R has an artinian total ring of quotients. Since hereditary, quasi-local, noetherian rings are prime rings, one could expect an affirmative answer to this question. For quasi-local noetherian rings with gl. dim $R \leqq 2$ whose Jacobson radical is a principal right ideal we can show that they are prime rings, because the following statement holds.

COROLLARY 2. *If R is a quasi-local noetherian ring with* gl. dim $R \leqq 2$ *whose Jacobson radical is a principal right ideal of R, then R is Morita equivalent to either a simple noetherian domain or to a quasi-local noetherian domain D with* gl. dim $D \leqq 2$.

PROOF. Since R is a quasi-local noetherian ring whose Jacobson radical J is a principal right ideal of R, the ring R is either semiprime or J is nilpotent by [**4**, Hilfssatz 4.1]. By Theorem 1 we may assume that J is nilpotent. If $J = nR$, then let $P = n_l = \{x \in R \mid xn = 0\}$. Since gl. dim $R \leqq 2$, P is a finitely generated projective left R-module. Therefore $\tau_R(P) = R$, because R is quasi-local. Hence

$$1 = p_1 f_1 + p_2 f_2 + \cdots + p_n f_n$$

for some $p_i \in P$ and some $f_i \in \mathrm{Hom}_R(P, R)$. If we had $P \leqq J$, then there would be a smallest positive integer k such that $P^k=0$. Since $k\neq 1$, there exists $0\neq y\in P^{k-1}$. Now

$$\begin{aligned} y &= y(p_1f_1) + y(p_2f_2) + \cdots + y(p_nf_n) \\ &= (yp_1)f_1 + (yp_2)f_2 + \cdots + (yp_n)f_n = 0, \end{aligned}$$

because $yp_i\in P^k=0$ for all i. This contradiction shows $P\nleqq J$. Hence $R=PR$ by Nakayama's Lemma. Thus

$$Rn = PRn \leqq P(nR) = (Pn)R = 0.$$

Therefore $J=0$, which implies that R is simple. Hence R is Morita equivalent to a simple noetherian domain D with gl. dim $R\leqq 2$ by Theorem 2, completing the proof of Corollary 2.

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Cornell University