FINITE GROUPS WITH PRO-NORMAL SUBGROUPS

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The purpose of this paper is to study a class of finite groups whose subgroups of prime power order are all pro-normal. Following P. Hall, we say that a subgroup H of a group G is pro-normal in G if and only if, for all x in G, H and H^x are conjugate in $\langle H, H^x\rangle$, the subgroup generated by H and H^x. We shall determine the structure of all finite groups whose subgroups of prime power order are pro-normal. It turns out that these groups are in fact soluble t-groups. A t-group is a group G whose subnormal subgroups are all normal in G. The structure of finite soluble t-groups has been determined by Gaschütz [1]. Are soluble t-groups precisely those groups whose subgroups of prime power order are all pro-normal? The answer is yes. Thus our study furnishes another characterization of soluble t-groups. Except for the definitions given above, our terminology and notation are standard (see, for example, M. Hall [2]).

We first prove two general facts about pro-normal subgroups.

LEMMA 1. *Let H be a pro-normal subgroup of a group G and let K be a subgroup of G containing H. Then H is subnormal in K if and only if H is normal in K.*

PROOF. Let H be subnormal in K. We may assume that there is a subgroup L of G such that H is normal in L and L is normal in K. Let x be any element of K. Then H^x lies in L and so $\langle H, H^x\rangle$ is a subgroup of L. Since H is pro-normal in G, there is an element y in $\langle H, H^x\rangle$ such that $H^y=H^x$. Since H is normal in L and y lies in L, $H^x=H$. Hence H is normal in K.

LEMMA 2. *Let G be a finite group. Let M be a normal p'-subgroup of G and let P be any p-subgroup of G. Then P is pro-normal in G if and only if PM/M is pro-normal in G/M.*

PROOF. Suppose P is pro-normal in G. Then for any element x in G, P and P^x are conjugate in $\langle P, P^x\rangle$ and so PM/M and P^xM/M are conjugate in $\langle P, P^x\rangle M/M=\langle PM/M, P^xM/M\rangle$. Hence PM/M is pro-normal in G/M.

Conversely, suppose PM/M is pro-normal in G/M. Then for any element xM in G/M, PM/M and P^xM/M are conjugate in $\langle P,P^x\rangle M/M$. Thus there exists an element y in $\langle P, P^x\rangle$ such that $P^yM/M=P^xM/M$.

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So $P^yM = P^xM$. Hence $\langle P^y, P^x \rangle$ is a subgroup of P^xM. Since P^x is a Sylow p-subgroup of P^xM, P^y and P^x are Sylow subgroups of $\langle P^y, P^x \rangle$. Hence there exists an element z in $\langle P^y, P^x \rangle$ such that $P^{yz} = P^x$. Now

$$\langle P^y, P^x \rangle \subseteq \langle P, P^x, y \rangle = \langle P, P^x \rangle.$$

Hence yz lies in $\langle P, P^x \rangle$ and so P and P^x are conjugate in $\langle P, P^x \rangle$.

THEOREM. *Let G be a finite group. Then all subgroups of prime power order of G are pro-normal in G if and only if G is a soluble t-group.*

PROOF. Suppose all subgroups of prime power order are pro-normal in G. We first prove that G is soluble by induction on the order of G. Let p be the smallest prime dividing the order of G. Let P be a Sylow p-subgroup of G and let Q be any subgroup of P. Suppose A is any cyclic subgroup of Q. Then A is subnormal in $N_G(Q)$, the normalizer of Q in G. Since A is pro-normal in G, A is normal in $N_G(Q)$ by Lemma 1. Let x be any p'-element of $N_G(Q)$. Then x induces an automorphism of A of order prime to p. Since the automorphism group of A is of order not divisible by any prime greater than p, x must in fact centralize A. Thus x centralizes Q. Let $C_G(Q)$ be the centralizer of Q in G. Then $N_G(Q)/C_G(Q)$ is a p-group. This holds for every subgroup Q of P. By a well-known result of Frobenius (see, for example, [2, Theorem 14.4.7]), G has a normal p-complement, K say. Clearly all subgroups of prime power order of K are pro-normal in K. Since K has smaller order than G, K is soluble by induction. Hence G is soluble.

We next show that G is in fact supersoluble. Let M be a normal p-subgroup of G for some prime p. Then every subgroup of prime power order of G/M is either a p-subgroup or a q-subgroup of the form QM/M for some q-subgroup Q of G, where q is a prime different from p. Hence all subgroups of prime power order of G/M are pro-normal in G/M by Lemma 2. This together with Lemma 1 shows that G is supersoluble.

We now prove that G is a t-group by induction on the order of G. Since G is supersoluble, G has a normal Sylow subgroup. Let P be a normal Sylow subgroup and let H be a subnormal subgroup of G. We may assume that there is a subgroup L of G such that H is normal in L and L is normal in G. Suppose $D = H \cap P \neq 1$. D is a normal subgroup of H and hence is subnormal in G. Since D has prime power order, D is normal in G. By induction G/D is a t-group. Hence H/D is normal in G/D and therefore H is normal in G. We may therefore assume that $H \cap P = 1$. Since P is a Sylow subgroup of G, H and P have coprime orders. Now G/P is also a t-group by induction. Hence

HP is normal in G. Thus $HP\cap L$ is normal in G. Since $HP\cap L = H(P\cap L)$ and H and $P\cap L$ have coprime orders, H is characteristic in $HP\cap L$ and hence is normal in G. Hence G is a t-group.

Conversely, suppose G is a soluble t-group. It is clear that G is actually supersoluble. Again we use induction on the order of G to prove that all subgroups of G of prime power order are pro-normal in G. Let P be a normal Sylow subgroup of G. Then all subgroups of P are normal in G and so are pro-normal in G. Let Q be a subgroup of prime power order not in P. Then since G/P is a soluble t-group, QP/P is pro-normal in G/P by induction. Since Q and P have coprime orders, Q is pro-normal in G by Lemma 2. Hence all subgroups of prime power order of G are pro-normal in G.

It is an immediate consequence of the theorem that subgroups of finite soluble t-groups are again t-groups. This is by no means obvious and was first proved by Gaschütz [1]. Again, let G be a finite group whose Sylow subgroups are all cyclic. It is clear that all subgroups of prime power order of G are pro-normal in G and so G is a t-group by our theorem. Thus we see that our theorem also gives a useful characterization of finite soluble t-groups.

References

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2. M. Hall, *The theory of groups*, Macmillan, New York, 1959.

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