# SELF-INJECTIVE SEMIGROUP RINGS FOR FINITE INVERSE SEMIGROUPS[1]

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The purpose of this article is the proof of the following theorem. $R$ will always denote a ring with identity, and $R(S)$ the semigroup ring (contracted if $S$ has a zero) of a semigroup $S$ over $R$.

THEOREM. *Let $S$ be a finite inverse semigroup. Then $R(S)$ is self-injective (s.i.) if, and only if, $R$ is s.i.*

This is an extension of Theorem 8.3 in [**3**], of part 1 of Theorem 4.1 in [**2**], and of the corollary to Theorem 1 in [**4**]. The results in [**2**] and [**3**] are used in its proof. As $R$ has an identity, $S$ is assumed embedded in $R(S)$. $|X|$ will denote the cardinality of the set $X$ and $X \setminus Y$ will denote the complement of a set $Y$ in a set $X$. Terminology and definitions are given in [**1**].

1. **An identity for $R(S)$ and the main theorem.** Let $S$ be an inverse semigroup, i.e. a regular semigroup in which idempotents commute. Let $E$ be the set of idempotents in $S$. Then $E$ is a commutative idempotent subsemigroup of $S$ and each principal left (right) ideal of $S$ has a unique idempotent generator [**1**, Theorem 1.17, p. 28]. Then $R(S)$ has an identity if $R(E)$ has one. If $Z$ denotes the ring of integers and $Z(E)$ has an identity, then $R(E)$ has an identity. Note that $E$ has a zero if $|E|$ is finite.

THEOREM 1. *If $E$ is a finite commutative idempotent semigroup, then $Z(E)$ has an identity.*

PROOF. The proof is by induction on $|E|$. If $|E|$ is 1 or 2, the result is clear. Since $|E|$ is finite there exists an element $u$ ($\neq 0$) in $E$ such that $uE = \{u, 0\}$. Then $I = \{u, 0\}$ is an ideal of $E$. $|E/I| < |E|$ and $E/I$, the Rees factor semigroup, is a commutative idempotent semigroup. Let $\phi': E \to E/I$ be the natural homomorphism and extend $\phi'$ linearly to the ring epimorphism $\phi: Z(E) \to Z(E/I)$ with kernel $Z(I)$. Let $\phi(a)$ be denoted by $\bar{a}$. $E$ and $E/I$ are assumed embedded in $Z(E)$ and $Z(E/I)$ respectively. Let $e^* = \sum_{a \in E \setminus I} \alpha(\bar{a})\bar{a}$, $\alpha(a) \in Z$, be the identity of $Z(E/I)$ insured by the induction hypothesis. Let $e' = \sum_{a \in E \setminus I} \alpha(a)a$

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$\in Z(E)$. Then $\phi(e')=e^*$ and $e^*\bar{x}=\bar{x}$ for $x\in E$ so $e'x\in x+Z(I)$. For $x\in E$, let $f(x)\in Z$ such that $e'x=x+f(x)u$. If $xu=u$ and $yu=u$, then $f(x)=f(y)$ since

$$(e'x)y = (x+f(x)u)y = xy + f(x)u$$

and

$$(e'y)x = (y+f(y)u)x = yx + f(y)u$$

and $E$ is commutative. Thus there is an $\alpha\in Z$ such that $f(x)=\alpha$ for each $x\in E$ such that $xu=u$. If $f(x)\neq 0$ there is a $w\in E$ such that $xw=u$ so $xu=u$ as $E$ is a semigroup of idempotents. Let $e=e'-\alpha u$. If $x\in E$, then $ex=e'x-\alpha ux=x+f(x)u-\alpha ux$. If $xu=u$, then $f(x)=\alpha$ and $ex=x$. If $xu=0$, then $f(x)=0$ and again $ex=x$. Hence, $Z(E)$ has an identity and the induction argument is complete.

This result can be extended somewhat. In what follows, $E$ and $E'$ will denote commutative idempotent semigroups. Let $M(E) = \{a: a\in E \text{ and } x\in E,\ ax=a \text{ imply } x=a\}$. Suppose $M(E)$ nonempty and $M(E)E=E$. Then for each $x\in E$ there is an $a\in M(E)$ for which $ax=x$. This condition is satisfied, for example, if in the set $P(E)$ of principal ideals of $E$, each chain has an upper bound (with respect to the partial ordering of $P(E)$ by set inclusion). If $E$ is finite $M(E)E=E$. *$M(E)E=E$ will be assumed for each idempotent semigroup in this article.* Clearly, $M(E)$ is contained in any set of generators of $E$. $E$ is said to be *unrefined* if $M(E)$ is a set of generators for $E$. A semigroup $E'$ is called a *refinement* of $E$ if $E$ is a subsemigroup of $E'$ and $M(E)=M(E')$. In general, $E'$ is a refinement of the subsemigroup $E$ generated by $M(E')$ and $E$ is an unrefined semigroup with $M(E)=M(E')$. If $e\in Z(E)$ is an identity for $Z(E)\subseteq Z(E')$ and $x\in E'$, there is an $a\in M(E')=M(E)\subseteq E$ such that $ax=x$. Then $ex=e(ax)=(ea)x=ax=x$, so $e$ is an identity for $Z(E')$ also. Thus, only unrefined semigroups need be considered in order to seek the existence of an identity. For example, let $E'=\{k: k \text{ a positive integer}\}$ and define $k*m=\max\{k, m\}$. Then $M(E')=\{1\}$ and $E'$ is a refinement of $E=\{1\}$.

LEMMA 1. *If $E$ is unrefined, then $E$ is finite if, and only if, $M(E)$ is finite.*

PROOF. If $|M(E)|=m$, $|E|\leqq 2^m-1$.

LEMMA 2. *If $M(E')$ is finite, then $E'$ is a refinement of a finite unrefined semigroup $E$.*

COROLLARY (TO THEOREM 1). *$Z(E)$ has an identity if, and only if, $M(E)$ is finite.*

PROOF. Assume $M(E)$ is finite and let $E'$ be the subsemigroup generated by $M(E)$. Then $E'$ is unrefined and $Z(E')$ has an identity by Lemma 1. Then $Z(E)$ has an identity by the remarks above as $M(E)E=E$.

Conversely, suppose $e=\sum_{i=1}^{p} n_i a_i$, $n_i \in Z$, $a_i \in E$, is an identity for $Z(E)$. Then it is an identity on $M(E)$. If there exists an $a \in M(E)$ with $a \neq a_i$ for each $i$, then $a_i a \neq a$ for each $i$ so $ea \neq a$; a contradiction. Thus, $M(E)$ is finite. (Note that if $M(E)$ is finite and $e$ is the identity of $Z(E)$, then each $a \in M(E)$ occurs with a nonzero coefficient in $e$.)

THEOREM 2. *Let $S$ be an inverse semigroup with idempotent semigroup $E$. Then $R(S)$ has an identity if, and only if, $M(E)$ is finite.*

PROOF. If $M(E)$ is finite, $Z(E)$ has an identity, say $e'=\sum_{i=1}^{p} n_i a_i$, $n_i \in Z$, $a_i \in E$. Let 1 be the identity of $R$ and let $e=\sum_{i=1}^{p}(n_i 1)a_i$. Then $e$ is an identity for $R(E)$. If $s \in S$, there are elements $a$ and $b$ in $E$ such that $as=s$ and $sb=s$ so $e$ is an identity for $R(S)$. Conversely, if $e=\sum_{i=1}^{p} r_i x_i$, $r_i \in R$, $x_i \in S$, is an identity for $R(S)$, then, in particular, $ea=a$ for each $a \in M(E)$. Thus $M(E)$ must be finite by the argument used in the proof of the corollary.

THEOREM 3. *Let $S$ be a finite inverse semigroup. Then $R(S)$ is s.i. if, and only if, $R$ is s.i.*

PROOF. Let $S=S_0 \supset S_1 \supset \cdots \supset S_{n+1}$ be a principal series for $S$ with $S_{n+1}=\{0\}$ if $S$ has a zero and $S_{n+1}$ empty otherwise. $S_i/S_{i+1}$ is a Brandt semigroup by [1, Exercise 3, p. 103], for each $i=0, 1, \cdots, n$. The proof is by induction on $n$. If $n=0$, $S \cong S_0/S_1$ is a Brandt semigroup so $S \cong M^0(G; m; m; \Delta)$, an $m \times m$ Rees matrix semigroup over a group with zero $G^0$ and with the $m \times m$ identity matrix $\Delta$ as a sandwich matrix [1, Theorem 3.9, p. 102]. Then $R(S) \cong M_m(R(G))$, the ring of $m \times m$ matrices over $R(G)$. $M_m(R(G))$ is s.i. if, and only if, $R(G)$ is s.i. by [3, Theorem 8.3]. As $G$ is finite, $R(G)$ is s.i. if, and only if, $R$ is s.i. by [2, Theorem 4.1].

As suppose $n>0$. Then $R(S/S_n)$ is s.i. if, and only if, $R$ is s.i. and $S/S_n$ is a finite inverse semigroup and has a principal series of length less than $n$. $S_n$ is a Brandt semigroup (so an inverse semigroup) so $R(S_n)$ $(\subseteq R(S))$ has an identity, say $f$. If $x \in R(S)$, $xf$ and $fx$ are in $R(S_n)$ so $xf=f(xf)=(fx)f=fx$ and $f$ is central in $R(S)$. Let $e$ be the identity of $R(S)$ insured by Theorem 2. Then $R(S)=R(S)(e-f) \oplus R(S)f$, a ring direct sum. $R(S)$ is s.i. if, and only if, both $R(S)(e-f)$ and $R(S)f$ are s.i. [4, Lemma 1]. $R(S)f=R(S_n)$ so $R(S)(e-f) \cong R(S)/R(S_n) \cong R(S/S_n)$. If $R$ is s.i., then $R(S/S_n)$ and $R(S_n)$ are s.i. so $R(S)$ is. If $R(S)$ is s.i., then $R(S_n)$ is s.i. so $R$ is s.i. as $S_n$ is a Brandt semigroup and the argument used in the case $n=0$ applies. This completes the induction.

Note that $M(E)$ finite is not sufficient to yield this result [**2**, Theorem 4.1].

A group $G$ is an inverse semigroup; so if $G$ is finite, $R(G)$ is s.i. if, and only if, $R$ is s.i. This is part of Connell's result. Utumi's result is that $M_m(R)$, the ring of $m \times m$ matrices over $R$, is s.i. if, and only if, $R$ is s.i. But $M_m(R)=R(S)$, where $S=\{e_{ij}: 1 \leqq i, j \leqq m\} \cup \{0\}$ with $e_{ij}e_{pq}=\delta_{jp}e_{iq}$, $\delta_{jp}$ the Kronecker delta, is an inverse semigroup.

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