A NOTE ON MAXIMAL LOCALLY COMPACT SEMIGROUPS

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1. **Introduction.** Let $\mathcal{S}$ be the class of semigroups, and let $\mathcal{L}$ be those members of $\mathcal{S}$ that are locally compact. We use the notation $S<T$ (S, $T\in\mathcal{S}$) to mean S is a subsemigroup of T. By $S\lhd T$ we mean both $S<T$ and S is dense in T. Finally, we introduce the notion of a maximal semigroup; an element S in $\mathcal{L}$ is said to be maximal in $\mathcal{L}(\mathcal{S})$ if S is closed in all $T\in\mathcal{L}(\mathcal{S})$ with $S<T$.

We first show that for each S in $\mathcal{L}$ there exists a T in $\mathcal{L}$ such that T is maximal in $\mathcal{L}$ and $S\lhd T$. Also, we show that there exists a noncompact $S\in\mathcal{L}$ that is maximal in $\mathcal{S}$. In fact, it will be shown that the semigroups described by Hoffman in [**1**] and those of the Mostert Type [**2**] are maximal in $\mathcal{S}$.

II. **Existence theorem.** In this section we prove the existence theorem and give most of the definitions and notation that will be used throughout this note.

If $S\lhd T$, then T will be called an M(S) *semigroup* if T is maximal in $\mathcal{S}$, and T will be called an LCM(S) *semigroup* if T is in $\mathcal{L}$ and T is maximal in $\mathcal{L}$.

For the existence theorem we will denote a semigroup by a triple (S, F, m), where S is the set, F is the topological structure and m is the multiplication function.

EXISTENCE THEOREM. *If* $(S, F, m)\in\mathcal{L}$, *then there exists an* LCM(S) *semigroup.*

PROOF. Let $d=\inf\{|N| \mid N\subseteq S,\ \overline{N}=S\}$, where $|N|$ denotes the cardinal number of N. We make use of the facts [**3**, pp. 50–52] that $|S|\leqq 2^{2^d}$ and if $S\lhd T$, then $|T|\leqq 2^{2^d}$.

Let Y be a set with $2^{2^d}<|Y|$ and $S\subseteq Y$, let X_1 be the set of subsets of Y, let X_2 be the set of subsets of X_1, let X_3 be the set of functions from $Y\times Y$ into Y, and finally, for f in $X_1\times X_2\times X_3$ let $\hat{f}(3)$ be $f(3)$ restricted to $f(1)\times f(1)$, where $f=(f(1), f(2), f(3))$.

Let C be the collection of all f in $X_1\times X_2\times X_3$ so that $f(3)(f(1)\times f(1))\subseteq f(1)$, $(f(1), f(2), \hat{f}(3))\in\mathcal{L}$ and $(S, F, m)\lhd(f(1), f(2), \hat{f}(3))$. Since there exists a g in $X_1\times X_2\times X_3$ with $g(1)=S$, $g(2)=F$ and

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$\hat{g}(3)=m$, $C\neq\emptyset$. Partially order C by: if f, l are in C, then $f\leqq l$ if and only if $(f(1), f(2), \hat{f}(3))<(l(1), l(2), \hat{l}(3))$.

Pick a maximal chain D from C, and let $R=\bigcup\{f(1)\mid f\in D\}$. Let $\mathfrak{A}=\{U\subseteq R\mid U\cap f(1)\in f(2)$ for all f in $D\}$. Define $n: R\times R\to R$ by $n(x, y)=f(3)(x, y)$ where $f\in D$ and f is such that $(x, y)\in f(1)\times f(1)$. By [4, p. 131], $(f(1), f(2))$ is an open subspace of $(R, \mathfrak{A})$ for each f in D. Since D is a chain and $(f(1), f(2))$ is open in $(l(1), l(2))$ whenever $f\leqq l$, $(R, \mathfrak{A})$ is locally compact and Hausdorff. Since $(R, \mathfrak{A})$ is a locally compact Hausdorff space, $\{U\subseteq R\times R\mid U\cap(f(1)\times f(1))$ is open in $f(1)\times f(1)$ for all f in $D\}$ is the product topology on $R\times R$. By [4, p. 132], n is continuous. Therefore, $(R, \mathfrak{A}, n)\in\mathfrak{L}$ and $(S, F, m)\lhd(R, \mathfrak{A}, n)$.

Let (T, F_2, p) be an element of $\mathfrak{L}$ with $(R, \mathfrak{A}, n)\lhd(T, F_2, p)$. Then, since $|T|\leqq 2^{2^d}<|Y|$ and $R\subset Y$, there exists a one-to-one function $\mathfrak{L}$ from T into Y so that $h|R$ is the identity map on R. Clearly, there exists a topological structure F_3 and a $p_1\in X_3$ so that $(R, \mathfrak{A}, n)<(h(T), F_3, \hat{p}_1)$ and $(h(T), F_3, p_1)\in\mathfrak{L}$. Let k be an element of C with $k(1)=h(T)$, $k(2)=F_3$ and $\hat{k}(3)=\hat{p}_1$. Since D is a maximal chain and $f\leqq k$ for all $f\in D$, $k\in D$. Thus $(k(1), k(2), \hat{k}(3))<(R, \mathfrak{A}, n)$ and $k(1)\subseteq R\subseteq k(1)=h(T)$. This completes the proof.

The following examples show that an LCM(S) semigroup need not be unique. (1) Let T_1 be the nonnegative real numbers with the usual topology and multiplication. In §3, we show T_1 is an LCM(S) semigroup for any $S\lhd T_1$. (2) In [5], Horne describes a locally compact semigroup with an open dense group P, where P is iseomorphic with the positive real numbers with the usual topology and ordinary multiplication and $S-P$ is not a compact group. By the existence theorem, there exists an LCM(S) semigroup T_2 and clearly, T_2 is also a LCM(P) semigroup.

III. M(S) **semigroups.** In this section we describe a collection of M(S) semigroups. In particular, we show why those semigroups in [1] and the ones of the Mostert Type [2] are M(S) semigroups.

THEOREM. *Let S be an element of $\mathcal{S}$, and suppose there exists a t in S that has a compact neighborhood. Let S be such that for each T in $\mathcal{S}$ with $S\lhd T$ there exists a group of homeomorphisms G acting on T with the following properties*: (1) $g(S)=S$ *for all* $g\in G$, *and* (2) *if U is an open set in T with $t\in U$ and $s\in T$, then there exists a $g\in G$ with $g(s)\in U$. Then $S\in\mathfrak{L}$ and S is a* M(S) *semigroup for any $S_1\in\mathcal{S}$ with $S_1\lhd S$.*

PROOF. Let T be an element of $\mathcal{S}$ with $S\lhd T$. Then, since S has a compact neighborhood at t, there exists a set U (open in T) so that $t\in U\subseteq S$.

Let x be an element of T. Then by (2) there exists an $h \in G$ so that $h(x) \in U \subseteq S$. By (1), $h^{-1}(h(x)) = x \in h^{-1}(S) = S$.

COROLLARY. *Let S be an element of $\mathcal{S}$ which contains* an *dense group G and a zero. If S has a compact neighborhood of zero, then $S \in \mathcal{L}$ and S is a* M(S_1) *semigroup for any S_1 in $\mathcal{S}$ with $S_1 \lhd S$.*

Using the above corollary and the fact [1] that a compact ideal can be regarded as a zero, we have the following corollary.

COROLLARY. *Let $S_1, \cdots, S_n$ be elements of $\mathcal{L}$ which contain dense groups $G_1, \cdots, G_n$ respectively and compact ideals $I_1, \cdots, I_n$ respectively. Then $\times_{i=1}^n S_i$ is a* M(S) *semigroup for any S in $\mathcal{S}$ with $S \lhd \times_{i=1}^n S_i$.*

THEOREM. *Let S be a* M(S) *semigroup with a* 1, *and let C be a compact group. Then $S \times C$ is a* M(S_1) *semigroup for any $S_1 \in \mathcal{S}$ with $S_1 \lhd S \times C$.*

PROOF. Let T be in $\mathcal{S}$ with $S \times C \lhd T$, and let $x \in T$. Then there exists a net $\{(g(\alpha), h(\alpha)) \mid \alpha \in A\}$ that is contained in $S \times C$ and converges to x. Since C is compact, there exists a g in C, a directed set B and a function θ from B into A so that $\{h(\theta(\beta)) \mid \beta \in B\}$ is a subnet of $\{h(\alpha) \mid \alpha \in A\}$ that converges to g. Then $\{(g(\theta(\beta)), h(\theta(\beta))) \mid \beta \in B\}$ converges to x and $\{(1, h(\theta(\beta))) \mid \beta \in B\}$ converges to $(1, g)$. Since C is a compact group, $\{(1, h(\theta(\beta))^{-1}) \mid \beta \in B\}$ converges to $(1, g^{-1})$. Therefore $\{(g(\theta(\beta)), h(\theta(\beta)))(1, h(\theta(\beta))^{-1}) \mid \beta \in B\}$ converges to $x(1, g^{-1})$. This implies $x(1, g^{-1}) \in \mathrm{Cl}(S \times \{1\}) = S \times \{1\}$. Therefore, $x \in S \times \{g\} \subseteq S \times C$. This completes the proof.

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