A REMARK ON A THEOREM OF A. WEIL

MORIKUNI GOTO[1]

1. The purpose of this paper is to prove the following theorem.

THEOREM. *Let G be a connected semisimple Lie group without compact components. Let H be a subgroup of G such that there exists a compact subset K of G with $G=HK$. Let σ be a continuous automorphism of G which reduces to the identity on H. Then σ is the identity automorphism of G.*

This is a generalization of a theorem of A. Weil in [**4**], and is related to a theorem of A. Borel in [**1**]. The author obtained the theorem by globalizing the infinitesimal method of Weil in [**4**].

2. Let G be a connected semisimple Lie group. Let $A(G)$ be the group of all continuous automorphisms of G. $A(G)$ is a Lie group with respect to the compact-open topology. Let $I(G)$ be the subgroup of $A(G)$ composed of all inner automorphisms. Then $I(G)$ is a closed normal subgroup of finite index in $A(G)$. For g in G we define $\mathrm{Ad}(g)$ by $\mathrm{Ad}(g)h=ghg^{-1}$ $(h\in G)$. Then $G\ni g\mapsto \mathrm{Ad}(g)\in I(G)$ gives a continuous homomorphism, whose kernel coincides with the center of G. Let $\mathfrak{G}$ be the Lie algebra of G, and let $\mathfrak{G}_1, \mathfrak{G}_2, \cdots, \mathfrak{G}_k$ be the simple factors of $\mathfrak{G}$: $\mathfrak{G}=\mathfrak{G}_1\oplus\mathfrak{G}_2\oplus \cdots \oplus\mathfrak{G}_k$ (direct sum of ideals). Let $I(\mathfrak{G})$, $I(\mathfrak{G}_1), \cdots$, and $I(\mathfrak{G}_k)$ be the adjoint groups of $\mathfrak{G}, \mathfrak{G}_1, \cdots$, and $\mathfrak{G}_k$ respectively. $I(\mathfrak{G})$ can be naturally identified with $I(G)$, and we have

$$I(G) = I(\mathfrak{G}) = I(\mathfrak{G}_1) \times I(\mathfrak{G}_2) \times \cdots \times I(\mathfrak{G}_k)$$

(direct product of closed normal subgroups). We denote by ϵ the identity automorphism of G.

LEMMA 1. *Let G be a connected semisimple Lie group. Let N be a nontrivial, i.e., $N\neq\{\epsilon\}$, normal subgroup of $A(G)$. Then there exists an i $(i=1, 2, \cdots$, or $k)$ with $N\supset I(\mathfrak{G}_i)$.*

PROOF. First suppose that $N\cap I(\mathfrak{G})=\{\epsilon\}$. Let σ be in N. For g in G we have $\mathrm{Ad}(g)\sigma=\sigma\mathrm{Ad}(g)$, which implies that $\sigma(g^{-1})g$ is in the center of G. On the other hand, $G\ni g\mapsto\sigma(g^{-1})g$ gives a continuous map from the connected space G. Since the center of G is discrete, we have

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$\sigma(g^{-1})g=$ the unit element for all g in G, i.e. $\sigma=\epsilon$. Hence $N=\{\epsilon\}$. This contradiction implies that $N\cap I(\mathfrak{G})\neq\{\epsilon\}$.

Since each $I(\mathfrak{G}_i)$ $(i=1, 2, \cdots, k)$ has no proper normal subgroup, i.e. $I(\mathfrak{G}_i)$ is a simple group, see Goto [**3**], any nontrivial normal subgroup of $I(\mathfrak{G})=I(\mathfrak{G}_1)\times I(\mathfrak{G}_2)\times\cdots\times I(\mathfrak{G}_k)$ contains at least one of the $I(\mathfrak{G}_i)$. Q.E.D.

3. Let L be a topological group. For a in L, we denote by $C(a)$ the conjugate class containing a. We define a subset $\mathfrak{C}(L)$ of L by the condition: $a\in\mathfrak{C}(L)$ if and only if the closure of $C(a)$ is compact. Then the following lemma holds obviously.

Lemma 2. *$\mathfrak{C}(L)$ is a normal subgroup of L.*

Proposition. *Let G be a connected semisimple Lie group without compact components. Then $\mathfrak{C}(A(G))=\{\epsilon\}$.*

Proof. If it is not true, then by Lemma 1 and Lemma 2 $\mathfrak{C}(A(G))$ must contain some $I(\mathfrak{G}_i)$. Hence it suffices to prove that $I(\mathfrak{G}_i)$ contains a closed subgroup M with $\mathfrak{C}(M)\neq M$.

Since $\mathfrak{G}_i$ is a noncompact simple Lie algebra, there exists a subalgebra $\mathfrak{M}$ of $\mathfrak{G}_i$, which is isomorphic with $\mathrm{sl}(2, \boldsymbol{R})$, the Lie algebra of all real 2 by 2 matrices with trace 0. (See Goto [**2**].) Since $I(\mathfrak{G}_i)$ is a Lie group composed of linear transformations, the analytic subgroup M of $I(\mathfrak{G}_i)$, corresponding to M, is closed and is isomorphic with $\mathrm{SL}(2, \boldsymbol{R})$, the real special linear group of two dimension, or with $I(\mathrm{SL}(2, \boldsymbol{R}))$. Since the conjugate class containing

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{SL}(2, \boldsymbol{R})$ contains all

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad (\alpha > 0),$$

we have $\mathfrak{C}(M)\neq M$. Q.E.D.

4. Proof of Theorem. Let σ be a continuous automorphism of G. Since $\sigma(h)=h$ implies that $\mathrm{Ad}(h)\sigma=\sigma\mathrm{Ad}(h)$, if $\sigma(h)=h$ for all h in H, then

$$\{\mathrm{Ad}(g)\sigma\mathrm{Ad}(g^{-1});\, g\in G\} = \{\mathrm{Ad}(k)\sigma\ \mathrm{Ad}(k^{-1});\, k\in K\}$$

is compact. Since $A(G)/I(G)$ is finite, $C(\sigma)$ is compact in $A(G)$. Hence $\sigma\in\mathfrak{C}(A(G))$, and by the Proposition $\sigma=\epsilon$. Q.E.D.

In a recent conversation with J. Tits, the author discovered that the main part of this paper is contained in J. Tits, *Automorphismes à deplacement borné des groupes de Lie*, Topology **3** (1964), 97–107.

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University of Pennsylvania