DENSITY OF ONE GRAPH ALONG ANOTHER[1]

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Introduction. The word "graph" shall mean the graph of a real function, and the X-projection of a graph F is the set of all abscissas of points of F. c denotes the cardinality of the continuum. If H is a collection of sets, H^* denotes the union of all the sets in H.

DEFINITION. Suppose F and G are graphs with X-projection $[0, 1]$. The statement that F is dense (c-dense) along G means that if $[a, b]$ is a subinterval of $[0, 1]$, then there is a point (are c-many points) of intersection of F and G with abscissa in $[a, b]$. In this paper, the following three theorems will be proved:

THEOREM 1. *If F is a graph with X-projection $[0, 1]$, then F is dense along the graph of a function of Baire class 1 with domain $[0, 1]$. However, there is a graph with X-projection $[0, 1]$ which is not dense along the graph of any continuous function with domain $[0, 1]$.*

THEOREM 2. *If F is a graph with X-projection $[0, 1]$, then F is c-dense along the graph of a Lebesgue measurable function with domain $[0, 1]$. However, there is a graph with X-projection $[0, 1]$ which is not c-dense along the graph of any Baire function with domain $[0, 1]$.*

THEOREM 3. *There exists a graph F with X-projection $[0, 1]$ which is c-dense along the graph of every Lebesgue measurable function with domain $[0, 1]$.*

Proofs. The author is indebted to the referee for the short proof of Theorem 1 which appears here. It makes use of a theorem of H. Blumberg and is considerably shorter than the author's original proof.

PROOF OF THEOREM 1. Suppose F is the graph of a function f with domain $[0, 1]$. It follows from Theorem III of [1] that there is a countable dense subset D of $[0, 1]$ such that $f|D$, the restriction of f to D, is continuous. Now, if $f|D$ is bounded, let g be defined as follows: $g(x)=f(x)$ if x is in D, and if x is in $[0, 1]-D$, $g(x)$ is the lim sup $f(t)$ as $t \to x$ with t in D. Then g is upper semicontinuous and is therefore in Baire class 1, and F is dense along the graph of g. Now if $f|D$ is not bounded, let $s_1, s_2, \cdots$ be a sequence of mutually exclusive segments such that D is a subset of $\bigcup s_j$ and $f|D$ is bounded on each s_j. For each

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positive integer j, let g_j be an upper semicontinuous function defined on s_j which agrees with f on $D \cap s_j$ and let g be the function defined as follows: $g(x) = g_j(x)$ if x is in s_j and if x is in $[0, 1] - \cup s_j$, $g(x) = 0$. The function g is in Baire class 1 on the open set $\cup s_j$ and on the closed set $[0, 1] - \cup s_j$. It is therefore in class 1 on all of $[0, 1]$, and F is dense along its graph.

The requirement of the second part of the theorem is satisfied by the graph of any function f with domain $[0, 1]$ such that for some t in $(0, 1)$, $f(t+)$ and $f(t-)$ both exist but are unequal.

PROOF OF THEOREM 2. Suppose F is the graph of a function f with domain $[0, 1]$. Let M be a subset of $[0, 1]$ with measure zero such that if $[a, b]$ is a subinterval of $[0, 1]$, then $[a, b] \cap M$ has cardinality c. Let $g(t) = f(t)$ if t is in M and $g(t) = 0$ if t is in $[0, 1] - M$. Clearly, g is Lebesgue measurable, and F is c-dense along its graph.

Now, let R, the set of all real numbers, be well ordered so that no element of R is preceded by c elements, and let T be a reversible transformation from $[0, 1]$ to the class of all Baire functions with domain $[0, 1]$. For each element X of $[0, 1]$, let f_X denote $T(X)$. Let h be the function defined as follows: for the first element X of $[0, 1]$, $h(X)$ is the first element of R different from $f_X(X)$, and if Y is an element of $[0, 1]$ such that $h(X)$ has been defined for every X in $[0, 1]$ which precedes Y, then $h(Y)$ is the first element of R not in $\{f_X(Y) \mid X$ is in $[0, 1]$ and X precedes $Y\}$. If F is the graph of a Baire function with domain $[0, 1]$ and H is the graph of h, then the set of *all* points of intersection of F and H has cardinality less than c. Therefore H is not c-dense along the graph of any Baire function with domain $[0, 1]$.

PROOF OF THEOREM 3. Every closed subset of $[0, 1]$ which has positive measure has a perfect subset (the set of its points of condensation) and is therefore of cardinality c. The collection of all such sets is of cardinality c, so from Theorem 2, p. 458, of [2] it follows that there is a collection W of c mutually exclusive subsets of $[0, 1]$, each of which has c elements in common with each closed subset of $[0, 1]$ of positive measure. Each set of W has outer measure one, or else there would be a closed subset of $[0, 1]$ of positive measure which some set of W would not intersect. Let T be a reversible transformation from W to the class of all Baire functions with domain $[0, 1]$ and for each set w of W, let g_w denote $T(w)$. Let f be the function with domain $[0, 1]$ defined as follows: if t belongs to a set w of W, $f(t) = g_w(t)$, and if t is in $[0, 1] - W^*$, $f(t) = 0$. Let F be the graph of f. Suppose G is the graph of a Lebesgue measurable function g with domain $[0, 1]$ and $[a, b]$ is a subinterval of $[0, 1]$. Let $[d, e]$ be a proper subinterval of $[a, b]$ and g' be a Baire function with domain

[0, 1] which agrees with g except on a set M of measure zero. Let g'' be that subset of g' which has domain $[d, e]$. g'' is a subset of c different Baire functions h with domain [0, 1], and for each of these a number t_h in $[d, e]$ shall be chosen such that $f(t_h)=g(t_h)$. Let h be a Baire function with domain [0, 1] of which g'' is a subset, and let w be $T^{-1}(h)$. Since w has outer measure one, there is a number t_h which is in $w\cap[d, e]$ but not in M. Thus $f(t_h)=h(t_h)=g''(t_h)=g'(t_h)=g(t_h)$. Since the sets in W are mutually exclusive, no two numbers t_h so chosen will be the same. Therefore, the set of points common to F and G which have abscissa in $[d, e]$ and therefore in $[a, b]$ has cardinality c.

References

1. H. Blumberg, *New properties of all real functions*, Trans. Amer. Math. Soc. **24** (1922), 113–128.

2. W. Sierpinski, *Cardinal and ordinal numbers*, 2nd rev. ed., Polska Akad. Nauk, Monogr. Mat. 34, PWN-Polish Scientific Publishers, Warsaw, 1965.

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