A COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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1. **Statement of results.** If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is analytic and univalent in the unit disk E ($|z|<1$), then it is known [1] that

$$
(1)\qquad \begin{aligned} |a_3-\mu a_2^2| &\leqq 4\mu-3 \quad \text{when } \mu\geqq 1,\\ &\leqq 1+2\exp[-2\mu/(1-\mu)] \quad \text{when } 0\leqq\mu\leqq 1,\\ &\leqq 3-4\mu \quad \text{when } \mu\leqq 0. \end{aligned}
$$

The result is sharp in the sense that for each μ there is a function in the class under consideration for which equality holds.

This paper contains analogues of (1) for certain classes of analytic functions. Explicitly, let γ and λ be real numbers, where $|\gamma|<\pi/2$ and $0\leqq\lambda<1$, and let $S(\gamma,\lambda)$ denote the class of analytic functions $f(z)$ in E such that $f(0)=0, f'(0)=1$ and

$$
(2)\qquad \operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\}>\lambda\cos\gamma \qquad (z\in E).
$$

In particular, $S(0,\lambda)$ is Robertson's class of functions that are starlike of order λ in E [6] and $S(0, 0)$ is the class of normalized starlike functions. The following sharp result is proved in §2.

THEOREM 1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is in $S(\gamma,\lambda)$ and if μ is a complex number, then*

$$
(3)\qquad |a_3-\mu a_2^2|\leqq(1-\lambda)\cos\gamma\max(1,|2\cos\gamma(1-\lambda)(2\mu-1)-e^{i\gamma}|).
$$

For each μ, there is a function in $S(\gamma,\lambda)$ for which equality holds.

Hummel ([2], [3]), using variational techniques, proves the conjecture of V. Singh that $|a_3-a_2^2|\leqq 1/3$ for the normalized convex functions in E. Since $zf'(z)$ is starlike if and only if $f(z)$ is convex in E [5, p. 223], the following extension of this result is obtained.

COROLLARY 1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is analytic and convex in E and if μ is a complex number, then $|a_3-\mu a_2^2|\leqq\max(1/3,|\mu-1|)$. The result is sharp for each μ.*

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A function $f(z)$ is spiral-like [7] in E if there is a real γ, $|\gamma| < \pi/2$, such that $f(z) \in S(\gamma, 0)$. Another simple consequence of Theorem 1 is

COROLLARY 2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is spiral-like in E and if μ is a complex number, then*

$$|a_3 - \mu a_2^2| \leqq 2|\mu - 1| + |2\mu - 1|.$$

For each real μ, there is a starlike function for which equality holds.

An analytic function $f(z) = z + \cdots$ in E is close-to-convex [4] if there is a real γ, $|\gamma| < \pi/2$, and a starlike function $g(z) = z + \cdots$ such that

$$\operatorname{Re}\left\{e^{i\gamma} \frac{zf'(z)}{g(z)}\right\} > 0 \qquad (z \in E). \tag{4}$$

In §3 we prove

THEOREM 2. *If the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in E is close-to-convex and if μ is a real number, then*

$$|a_3 - \mu a_2^2| \leqq \max(1, 3|\mu - 1|, |4\mu - 3|). \tag{5}$$

If μ is outside the interval $(0, 2/3)$, there is an analytic close-to-convex function for which equality holds.

Let K_0 be the subclass of analytic close-to-convex functions $f(z)$ such that (4) holds with $\gamma = 0$ for some starlike function $g(z) = z + \cdots$ in E. In §4 we prove the following sharp result.

THEOREM 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in K_0 and if μ is real, then*

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqq 3 - 4\mu \quad \text{for } \mu \leqq 1/3, \\ &\leqq 1/3 - 4/9\mu \quad \text{for } 1/3 \leqq \mu \leqq 2/3, \\ &\leqq 1 \quad \text{for } 2/3 \leqq \mu \leqq 1, \\ &\leqq 4\mu - 3 \quad \text{for } \mu \geqq 1. \end{aligned} \tag{6}$$

For each μ, there is a function in K_0 such that equality holds.

We suspect that the bounds in (6) are sharp when $\mu \in (0, 2/3)$ for the wider class of all analytic close-to-convex functions.

2. **Proof of Theorem 1.** First, if $\Phi(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ is in the class B of functions that are analytic in E and map the unit disk into itself, then $|\alpha_2| \leqq 1 - |\alpha_1^2|$ (for example, see [5, p. 108]). Therefore, if s is a complex number, we have

$$
\begin{aligned}
|\alpha_2 - s\alpha_1^2| &\leqq |\alpha_2| + |s|\,|\alpha_1^2| \leqq 1 + (|s| - 1)|\alpha_1^2| \\
&\leqq \max(1, |s|).
\end{aligned} \tag{7}
$$

Moreover, the functions $\Phi(z)=z$ and $\Phi(z)=z^2$ respectively show that the result is sharp for $|s|\geqq 1$ and for $|s|<1$. Now, by (2), $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ is in $S(\alpha, \lambda)$ if and only if the function

$$
\Phi(z) = \frac{f'(z) - f(z)/z}{f'(z) + [(1-\lambda)e^{-2i\gamma} - \lambda]f(z)/z} = \sum_{n=1}^{\infty} \alpha_n z^n
$$

is in the class B. A simple computation shows

$$
\alpha_1 = \frac{u a_2}{1-\lambda}, \quad \alpha_2 = \frac{2u}{1-\lambda}\left[a_3 - \frac{1-\lambda-u}{2(1-\lambda)} a_2^2\right], \quad u = \frac{e^{i\gamma}}{2\cos\gamma}. \tag{8}
$$

The inequality (3) with

$$
\mu = \frac{1-\lambda + (s+1)u}{2(1-\lambda)}
$$

is now obtained by substituting the coefficients (8) into (7). That (3) is sharp follows from the sharpness of the inequalities (7).

REMARK. The same argument also proves

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\leqq (1-\lambda)\cos\gamma + (|2\cos\gamma(1-\lambda)(2\mu-1) - e^{i\gamma}| - 1) \\
&\quad \cdot |a_2^2| / 4(1-\lambda)\cos\gamma.
\end{aligned}
$$

For each a_2, where $|a_2| < 2(1-\lambda)\cos\gamma$, and for each complex number μ, there is a function in $S(\gamma, \lambda)$ for which equality holds.

3. **Proof of Theorem 2.** By (4) the analytic function $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ in E is close-to-convex if and only if there exists a $g(z) = z+\sum_{n=2}^{\infty} c_n z^n$ in $S(0, 0)$ such that the function

$$
\Phi(z) = e^{i\gamma}\frac{f'(z) - g(z)/z}{f'(z) + e^{-2i\gamma} g(z)/z} = \sum_{n=1}^{\infty} \alpha_n z^n
$$

is in the class B of §2. A comparison of the coefficients in the various power series expansions for the functions in this identity shows

$$
2a_2 = c_2 + 2\cos\gamma\alpha_1, \qquad 3a_3 = c_3 + 2\cos\gamma(\alpha_1 c_2 + \alpha_2 + e^{i\gamma}\alpha_1^2).
$$

Therefore, we have

$$
\begin{aligned}
a_3 - \mu a_2^2 &= \tfrac{1}{3}(c_3 - \tfrac{3}{4}\mu c_2^2) + \tfrac{2}{3}\cos\gamma[\alpha_2 + (e^{i\gamma} - \tfrac{3}{2}\mu\cos\gamma)\alpha_1^2] \\
&\quad + (\mu - \tfrac{2}{3})\cos\gamma\alpha_1 c_2.
\end{aligned} \tag{9}
$$

Set $\mu=2/3$. By (7) and Theorem 1, we obtain

$$\begin{aligned} |a_3-\tfrac{2}{3}a_2^2| &\leqq \tfrac{1}{3}|c_3-\tfrac{1}{2}c_2^2| + \tfrac{2}{3}\cos\gamma\,|\alpha_2+i\sin\gamma\alpha_1^2| \\ &\leqq \tfrac{1}{3}+\tfrac{2}{3}\cos\gamma \leqq 1. \end{aligned}$$

From the Area Theorem [**5**, p. 210], we have $|a_3-a_2^2|\leqq 1$ and by (9), we get $|a_3|\leqq 3$. Thus for $0\leqq\mu\leqq 2/3$, it follows that

$$|a_3-\mu a_2^2| \leqq \tfrac{3}{2}\mu|a_3-\tfrac{2}{3}a_2^2| + (1-\tfrac{3}{2}\mu)|a_3| \leqq 3(1-\mu)$$

and, for $2/3\leqq\mu\leqq 1$, that

$$|a_3-\mu a_2^2| \leqq (3\mu-2)|a_3-a_2^2| + 3(1-\mu)|a_3-2a_2^2/3| \leqq 1.$$

The last result is sharp since the close-to-convex class include the starlike functions $S(0, 0)$ and the inequality is sharp in the latter class by Theorem 1. Finally, if μ is not in the interval $[0, 1]$, then by (1) $|a_3-\mu a_2^2|\leqq|4\mu-3|$ since the close-to-convex functions are univalent [**4**].

4. **Proof of Theorem 3.** From (7), (9) with $\gamma=0$ and Theorem 1 for the starlike class, we have

$$\begin{aligned} |a_3-\mu a_2^2| \leqq\ & \tfrac{1}{3}\{1+\tfrac{1}{4}[|3\mu-3|-1]|c_2^2|\} \\ &+\tfrac{2}{3}\{1+\tfrac{1}{2}[|3\mu-2|-2]|\alpha_1^2|\} \\ &+\tfrac{1}{3}|3\mu-2|\,|\alpha_1|\,|c_2|. \end{aligned}$$

If $1/3\leqq\mu\leqq 2/3$, this becomes

$$\begin{aligned} |a_3-\mu a_2^2| &\leqq 1+\frac{1}{12}\{(2-3\mu)|c_2^2| \\ &\qquad + 4(2-3\mu)|\alpha_1|\,|c_2| - 12\mu|\alpha_1^2|\} \\ &= 1+\frac{1}{12}\left\{2-3\mu+\frac{(2-3\mu)^2}{3\mu}\right\}|c_2^2| \\ &\qquad -\mu\left\{|\alpha_1|-\frac{(2-3\mu)}{6\mu}|c_2|\right\}^2 \\ &\leqq 1+\frac{2-3\mu}{18\mu}|c_2^2| \leqq \frac{1}{3}+\frac{4}{9\mu}, \end{aligned}$$

since $|c_2|\leqq 2$. The result is sharp since there is a starlike function (the Koebe function $g(z)=z/(1-z)^2$) with $c_2=2$, $c_3=3$ and a function in B with $\alpha_1=(2-3\mu)/3\mu$, $\alpha_2=1-\alpha_1^2$, provided $1/3\leqq\mu\leqq 2/3$. For $0\leqq\mu\leqq 1/3$, we have

$$|a_3 - \mu a_2^2| \leqq 3\mu |a_3 - a_2^2/3| + (1 - 3\mu)|a_3| \leqq 3 - 4\mu.$$

For the remaining choices of μ, (6) is a consequence of Theorem 2. The sharpness for μ not in the interval (1/3, 2/3) follows from Theorem 1, since $S(0, 0) \subset K_0$.

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