A UNIQUENESS THEOREM FOR SOME NONLINEAR BOUNDARY VALUE PROBLEMS[1]

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In an earlier paper, it was shown, using a modified energy integral technique, that the Dirichlet and Neumann boundary value problems for linear differential equations of a certain class always have a unique solution [1]. In this paper, a theorem is proved which extends the linear uniqueness theorem to some nonlinear boundary value problems.

In the following the domains of the variable $x=(x_1, x_2, \cdots, x_n)$ and the parameter z are taken to be X and D respectively. Their boundaries are written ∂X and ∂D. A subdomain of D, either proper or improper, is denoted by D^*. The set $U_F(\lambda, \mu)$ is defined to be the collection of pairs of complex numbers $(\lambda_\alpha, \mu_\alpha)$ which are such that

(i) $F(x, z, \lambda_\alpha, \mu_\alpha)$ is analytic in z on $D\times X$ and

(ii) there exist two real valued functions $p_\lambda(x, z)$ and $p_\mu(x, z)$ defined on $X\times\partial D$ such that

$$\begin{aligned} &\left| F(x, z, \lambda_\alpha, \mu_\alpha) - F(x, z, \lambda, \mu) \right| \\ &< p_\lambda(x, z)\left| \lambda_\alpha - \lambda \right| + p_\mu(x, z)\left| \mu_\alpha - \mu \right| . \end{aligned} \tag{1}$$

In the following, the function $v(x, z_0)$ $(z_0\in D^*)$ will be said to be $F^{(u)}$-admissible if $(v, \nabla_x v)\in U_F(u, \nabla u)$ and both v and $\nabla_x v$ are square integrable $(\nabla_x=(\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n))$. In the complex number plane, half planes whose boundaries contain the origin shall be denoted by H. The outward drawn unit normal to such a half plane is v_H.

The operator

$$L = -\nabla_x\cdot a(x, z)\nabla_x + b(x, z) \tag{2}$$

with $a(x, z)$ and $b(x, z)$ analytic in z on $D\times X$ is of class H_0 if and only if

(i) for each $z\in\partial D$ and every pair of real numbers $(\xi, \eta)\neq(0, 0)$, there exists an H such that the mapping $h_{ab}(\xi, \eta)$ of X

$$h_{ab}(\xi, \eta) = a(x, z)\xi^2 + b(x, z)\eta^2 \tag{3}$$

is contained in the interior of H, and

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(ii) the winding number (with respect to the origin of the complex plane) of v_H, defined as a continuous function of z on ∂D, is zero.

Consider the equation

$$L[u] = F(x, z, u, \nabla_x u). \tag{4}$$

Problem N will be to solve equation (4) on $D^* \times X$ together with a Neumann condition on $D^* \times \partial X$. Problem D will be to solve equation (4) on $D^* \times X$ together with a Dirichlet condition on $D^* \times \partial X$.

THEOREM. *Assume that $L \in H_0$. Assume that on $X \times \partial D$ there exists a real function $\theta(z)$, which is such that*

$$\tfrac{1}{2}p_\mu(x, z) \leqq \min_{x \in X} \operatorname{Re}\{a(x, z)e^{-i\theta(z)}\},$$

$$\tfrac{1}{2}p_\mu(x, z) + p_\lambda(x, z) \leqq \min_{x \in X} \operatorname{Re}\{b(x, z)e^{-i\theta(z)}\}.$$

Then any $F^{(u)}$-admissible function $v(x, z)$ which solves either problem N *or problem* D *is unique in $U_F(u, \nabla u)$ if u is a solution.*

PROOF. Assume that an $F^{(u)}$-admissible function u solves problem N (or D) and that there exists a second $F^{(u)}$-admissible solution u_0 for some value of $z_0 \in D^*$. Then their difference $\psi(x, z_0) = u_0(x, z_0) - u(x, z_0)$ solves the equation

$$L(\psi) = F(x, z, u_0, \nabla_x u_0) - F(x, z, u, \nabla_x u) = \mathfrak{F}(x, z, u_0, u) \tag{5}$$

when $z = z_0$, together with appropriate data equal to zero on $D^* \times \partial X$. Therefore, multiplying equation (6) by $\bar{\psi}(x, z_0)$, the complex conjugate of $\psi(x, z_0)$, and integrating by parts yields

$$0 = \int_X dx[\{h_{ab}(|\psi|, |\nabla_x \psi|)\} - \bar{\psi}\mathfrak{F}(x, z, u_0, u)]. \tag{6}$$

Define the function

$$\begin{aligned} S(z, z_0) &= \int_X dx\{a(x, z)|\nabla_x \psi(x, z_0)|^2 + b(x, z)|\psi(x, z_0)|^2\} \\ &\quad - \int_X dx\{\bar{\psi}(x, z_0)\mathfrak{F}(x, z, u_0(x, z_0), u(x, z_0))\}. \end{aligned} \tag{7}$$

Since u_0 and u are admissible, $S(z, z_0)$ is analytic in z on D. From equation (6) we conclude that $S(z_0, z_0) = 0$. From equation (1), the absolute value of the second integral is less than

$$\int_x dx[\{p_\lambda(x, z) + \tfrac{1}{2}p_\mu(x, z)\}|\psi(x, z_0)|^2 + \{\tfrac{1}{2}p_\mu(x, z)\}|\nabla_x \psi(x, z_0)|^2]$$

which is less than the absolute value of the first integral on $X \times \partial D$. By Rouché's theorem, the number of zeros of $S(z, z_0)$ and the number of zeros of the first integral in D are the same. However, it has already been shown in an earlier paper that if $L \in H_0$, the first integral cannot vanish for any value of $z \in D$ when $\psi(x, z)$ is the difference of two admissible functions, unless $\psi(x, z_0)=0$ almost everywhere [1]. Thus, the solution is unique.

Reference

1. A. Kadish, J. Math. Phys. 9 (1968), 1266.

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