A NOTE ON THE LITTLEWOOD-TAUBER THEOREM

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The application of Wiener's generalized Tauberian Theorem [1] to a sequence of functions in $L_1(-\infty, \infty)$ leads to the following.

LEMMA. *Let* K_0 *belong to* $L_1(-\infty, \infty)$ *and be such that*

$$\int_{-\infty}^{\infty} K_0(x)e^{iux}dx \neq 0, \qquad -\infty < u < \infty.$$

Let g *be bounded on* $(-\infty, \infty)$ *and let*

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} K_0(x-y)g(y)dy = A\int_{-\infty}^{\infty} K_0(y)dy.$$

Then if $\{K_n(x)\}$ *is a sequence of functions in* L_1 *such that for* $n\to\infty$

$$\int_{-\infty}^{\infty} K_n(x-y)g(y)dy \to I(x)$$

for almost all x *and*

$$\int_{-\infty}^{\infty} K_n(y)dy \to I,$$

we have

$$\lim_{x\to\infty} I(x) = AI.$$

As an application of this lemma we give a particularly simple proof of the Littlewood-Tauber Theorem [2].

THEOREM. *Let* $\sum_0^\infty a_nx^n$ *converge to* $f(x)$ for $|x| < 1$ *and let*

$$\lim_{x\to 1-} f(x) = s < \infty.$$

If $n|a_n| < K < \infty$, *then* $\sum_0^\infty a_n = s$.

PROOF.

$$\left|\sum_0^N a_n - \sum_0^\infty a_ne^{-n/N}\right| \leqq \sum_0^N |a_n|(1-e^{-n/N}) + \sum_{N+1}^\infty |a_n| e^{-n/N}$$

$$\leqq K\left(2 + \sum_{N+1}^\infty \frac{e^{-n/N}}{n}\right) < \infty \qquad \text{for } N > 1,$$

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hence $s(u) = \sum_0^{[u]} a_n$ is bounded. Now,

$$f(e^{-x}) = x \int_0^\infty e^{-ux} s(u) du$$
$$= \int_0^\infty e^{-(\xi-y)} e^{-e^{-(\xi-y)}} s(e^y) dy,$$

where $x = e^{-\xi}$. Thus,

$$s \equiv \lim_{x \to 0^+} f(e^{-x}) = \lim_{\xi \to \infty} \int_{-\infty}^\infty K_0(\xi - y) s(e^y) dy,$$

where $K_0(x) = e^{-x} e^{-e^{-x}}$. We have

$$\int_{-\infty}^\infty K_0(x) e^{iux} dx = \Gamma(1 - iu) \neq 0.$$

Let $K_n(x) = (n/\pi)^{1/2} e^{-nx^2}$, then the conditions of the Lemma are satisfied under the hypothesis that $n|a_n|$ is bounded and, therefore, noting that in this case $I(x) = g(x)$, $I = 1$, we have

$$s = \lim_{x \to \infty} \lim_{n \to \infty} \int_{-\infty}^\infty K_n(x - y) s(e^y) dy = \lim_{x \to \infty} s(e^x) = \sum_0^\infty a_n.$$

References

1. N. Wiener, *The Fourier integral*, Dover, New York, 1933; p. 73.
2. G. H. Hardy, *Divergent series*, Oxford Univ. Press, New York, 1949; p. 154.

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