SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

TOEPLITZ-HAUSDORFF THEOREM ON NUMERICAL RANGES

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We give here a simple new proof of the following well-known

THEOREM. *Let $\mathfrak{X}$ be an inner product space over the field C of all complex numbers (defined as in* [1, *p.* 106]), *and let A be a linear operator on $\mathfrak{X}$ into itself. Then the numerical range of A, that is the set*

$$W = \{(AX, X): X \in \mathfrak{X} \text{ and } \|X\| = 1\},$$

will be a convex subset of the complex plane.

PROOF. We need consider only the case where the set W contains at least two points. Let X_k ($k=1, 2$) be any two elements of $\mathfrak{X}$ with $\|X_k\|=1$ such that $(AX_k, X_k)=w_k$ are two distinct points of W. As $X_1+zX_2=0$ for a z in C will imply that $|z|=1$ and then that $w_1=w_2$, we see that $\|X_1+zX_2\|\neq 0$ for all z in C. So the theorem will be proved if we show that, for any given real number t with $0<t<1$, there exists at least one complex number $z=x+iy$ (with x, y real) which satisfies the equation

$$(1)\quad (A(X_1+zX_2), X_1+zX_2) = (tw_1+(1-t)w_2)\cdot(\|X_1+zX_2\|)^2.$$

This equation may be rewritten in the form

$$(2)\qquad p\cdot|z|^2+q\cdot z+r\cdot\bar{z}+s=0,$$

where $p=t(w_2-w_1)$, $s=(1-t)(w_1-w_2)$ and q, $r\in C$. Dividing this equation by p, and then separating the real and imaginary parts, we get the two equations

$$(3)\qquad x^2+y^2+ax+by-((1-t)/t)=0,$$

and

$$(4)\qquad cx+dy=0,$$

where a, b, c, d are some well-defined real numbers such that this pair of equations is equivalent to the single equation (1).

On the rectangular cartesian (x, y)-plane, the equation (3) repre-

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sents a real circle with a positive radius having the origin in its interior (because the constant term in this equation is negative); and when c, d are not both zero, the straight line represented by the equation (4) meets this circle in two real and distinct points. We can, therefore, always find (at least) two distinct complex numbers z_k such that $z=z_k$ satisfy the equation (1). This proves slightly more than what we set out to prove.

For real inner product spaces $\mathfrak{X}$ we may similarly reduce the proof of the corresponding theorem to showing that a certain quadratic equation with real coefficients has real roots.

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Reference

1. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York 1958.

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A NOTE ON ABSOLUTE SUMMABILITY[1]

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Let A be an infinite matrix defining a sequence to sequence mapping by $(Ax)_n = \sum_k a_{nk}x_k$. The purpose of this note is to present a short elementary proof of the result that characterizes l-l methods (if $\sum_k |x_k|$ converges, then $\sum_n |(Ax)_n|$ converges). The proof in [2] is complicated by the fact that A is applied to the sequence of partial sums, rather than to x itself. Although the proof of Knopp and Lorentz [1] is elegant, it depends on the Principle of Uniform Boundedness.

Theorem. *The matrix A defines an l-l method if and only if there is a number M such that for each k*

$$(*) \qquad \sum_n |a_{nk}| \leqq M.$$

Proof. The sufficiency of (*) is easy since it yields

$$\sum_n |(Ax)_n| \leqq M \sum_k |x_k|.$$

If A is an l-l method, it is clear that each row sequence of A must be bounded: say $|a_{nk}| \leqq B_n$ for each k. It is also obvious that each

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