THE OPERATOR EQUATION $BX-XA=Q$ WITH SELFADJOINT A AND B[1]

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1. **Introduction.** Suppose A and B are possibly unbounded self-adjoint operators and Q is a bounded operator on a separable complex Hilbert space $\mathfrak{H}$. We will be concerned with the operator equation

(**E**) $BXf-XAf=Qf$ for all f in the domain $\mathfrak{D}(A)$ of A, where X is a bounded operator on H.

Our work is a continuation of the papers [**7**], [**4**]. There it is proved that if a, b and q are elements in a complex Banach algebra $\mathfrak{A}$ with identity e such that a and b have disjoint spectra, then the equation $bx-xa=q$ has a unique solution x in $\mathfrak{A}$, with

$$x = \frac{1}{2\pi i}\int_C (b - ze)^{-1}q(a - ze)^{-1}dz. \tag{1}$$

Here C is the boundary of a Cauchy domain containing the spectrum of a in its interior and disjoint from the spectrum of b. This result, when applied to problem (**E**), treats only the restrictive situation where A and B are bounded and have disjoint spectra. Thus the commutator equation where $A=B$ and other interesting cases are excluded. See Putnam [**6**, pp. 13–14] for further discussion.

We shall treat problem (**E**) by a perturbation procedure. If y is a positive real number, then $B+iyI$ and $A-iyI$ have disjoint spectra. We find that the equation $(B + iyI)T_yf-T_y(A-iyI)f=Qf, f\in\mathfrak{D}(A)$, has a unique solution T_y. (**E**) has a bounded solution X if and only if $\{\|T_y\|: y>0\}$ is bounded. In this case a sequence $\{T_{y_n}\}$ with $y_n\to 0$ converges weakly to a solution X of (**E**). T_y is explicitly given in terms of A, B and Q.

2. **Solution of** $BX-XA=Q$. It is clear that a bounded operator X is a solution of problem (**E**) if and only if it is a solution of problems (**E′**) or (**E″**):

(**E′**) $\langle Xf, Bg\rangle-\langle XAf, g\rangle=\langle Qf, g\rangle$ for all $f\in\mathfrak{D}(A)$ and $g\in\mathfrak{D}(B)$;

$$\begin{aligned}X(A - zI)^{-1} - (B - z^*I)^{-1}X &= (B - z^*I)^{-1}Q(A - zI)^{-1}\\ &\quad + 2iy(B - z^*I)^{-1}X(A - zI)^{-1},\end{aligned} \tag{E″}$$

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where $z=x+iy$, $z^*=x-iy$, x, y real, $y\neq 0$.

Our main theorem is prefaced by three lemmas. The first is essentially contained in Lengyel [3].

LEMMA 1. *Suppose A is selfadjoint. If f or g is in $\mathfrak{D}(A)$ and $y\neq 0$, then*

$$\lim_{\alpha\to\infty}\frac{1}{\pi i}\int_{-\alpha}^{\alpha}\langle (A-zI)^{-1}f,\, g\rangle dx = \operatorname{sgn} y\langle f, g\rangle. \tag{2}$$

PROOF. Set $R(z)=z^{-1}\langle f, g\rangle+\langle (A-zI)^{-1}f, g\rangle$. If $f\in\mathfrak{D}(A)$, then $R(z)=z^{-1}\langle (A-zI)^{-1}Af, g\rangle$ and $|zR(z)|\leqq |y|^{-1}\|Af\|\,\|g\|$. Thus if $f\in\mathfrak{D}(A)$ or $g\in\mathfrak{D}(A)$ there exists a constant M such that $|zR(z)| \leqq |y|^{-1}M$. This condition forces

$$\lim_{\alpha\to\infty}\frac{1}{\pi i}\int_{-\alpha}^{\alpha}R(z)dx = 0.$$

(See [3].) Thus

$$\lim_{\alpha\to\infty}\frac{1}{\pi i}\int_{-\alpha}^{\alpha}\langle (A-zI)^{-1}f, g\rangle dx = -\lim_{\alpha\to\infty}\frac{1}{\pi i}\int_{-\alpha}^{\alpha}z^{-1}\langle f, g\rangle dx$$
$$= \operatorname{sgn} y\langle f, g\rangle.$$

For A and B selfadjoint, Q is bounded, and $y>0$ we define the operator $T_y=T(A, B, Q, y)$ by the weak integral

$$T_y = \frac{1}{2\pi i}\int_{-\infty}^{\infty}(B-z^*I)^{-1}Q(A-zI)^{-1}dx. \tag{3}$$

LEMMA 2. $\|T_y\|\leqq (2y)^{-1}\|Q\|$.

PROOF. Suppose $f, g\in\mathfrak{H}$. Then

$$\int_{-\infty}^{\infty}\left|\langle (B-z^*I)^{-1}Q(A-zI)^{-1}f, g\rangle\right| dx$$
$$\leqq \|Q\|\int_{-\infty}^{\infty}\|(A-zI)^{-1}f\|\,\|(B-zI)^{-1}g\|dx$$
$$\leqq \|Q\|\left[\int_{-\infty}^{\infty}\|(A-zI)^{-1}f\|^2dx\right]^{1/2}\left[\int_{-\infty}^{\infty}\|(B-zI)^{-1}g\|^2dx\right]^{1/2}.$$

Now, if E_t is the resolution of the identity of A, then

$$\begin{aligned}\int_{-\infty}^{\infty}\|(A-zI)^{-1}f\|^2dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|t-z|^{-2}d\langle E_tf,f\rangle\right]dx\\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|t-z|^{-2}dx\right]d\langle E_tf,f\rangle\\ &= \pi y^{-1}\|f\|^2.\end{aligned}$$

Thus the form b defined on $\mathfrak{H}\times\mathfrak{H}$ by

$$b(f,g)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle(B-z^*I)^{-1}Q(A-zI)^{-1}f,g\rangle dx$$

is bounded with $\|b\|\leqq(2y)^{-1}\|Q\|$, and so T_y is well defined by (3) and $\|T_y\|=\|b\|$.

LEMMA 3. *T_y as given by (3) is the unique bounded solution of the operator equation*

$$(B+iyI)T_yf-T_y(A-iyI)f=Qf \qquad \textit{for all } f\in\mathfrak{D}(A). \tag{4}$$

PROOF. Suppose (4) has a bounded solution T. Then for all $f\in\mathfrak{D}(A)$, $g\in\mathfrak{D}(B)$, and $y>0$

$$\langle T(A-zI)^{-1}f,g\rangle-\langle Tf,(B-zI)^{-1}g\rangle=\langle(B-z^*I)^{-1}Q(A-zI)^{-1}f,g\rangle.$$

We apply $\lim_{\alpha\to\infty}(1/2\pi i)\int_{-\alpha}^{\alpha}\cdots dx$ to both sides of this last equation and employ Lemma 1 to deduce that $\langle Tf,g\rangle=\langle T_yf,g\rangle$, so $T=T_y$. Thus if a bounded solution to (4) exists, it is equal to T_y.

We conclude by showing that T_y is a solution. If $f\in\mathfrak{D}(A)$, and $g\in\mathfrak{D}(B)$,

$$\begin{aligned}&\langle T_yf,(B-iyI)g\rangle-\langle T_y(A-iyI)f,g\rangle\\ &\quad=\lim_{\alpha\to\infty}\frac{1}{2\pi i}\int_{-\alpha}^{\alpha}[\langle Q(A-zI)^{-1}f,g\rangle-\langle(B-z^*I)^{-1}Qf,g\rangle]dx,\end{aligned}$$

which by Lemma 1 equals $\langle Qf,g\rangle$. This implies (4).

Our main result comes next.

THEOREM 1. **(E)** *has a bounded solution if and only if $\{\|T_y\|:y>0\}$ is bounded.*

If $\{\|T_y\|:y>0\}$ is bounded then there exists a weakly convergent sequence $\{T_{y_n}\}$ with $y_n\to 0$. Any such weakly convergent sequence converges weakly to a bounded solution of **(E)**.

PROOF. Suppose **(E)** has a bounded solution X, and $f\in\mathfrak{D}(A)$,

$g\in\mathfrak{D}(B)$, $y>0$. Upon applying $\lim_{\alpha\to\infty}(1/2\pi i)\int_{-\alpha}^{\alpha}\cdots dx$ to the equation in ($\mathbf{E}''$) we obtain

$$\langle Xf, g\rangle = \langle T(A, B, Q, y)f, g\rangle + 2iy\langle T(A, B, X, y)f, g\rangle.$$

However, Lemma 2 implies that $\{y\|T(A, B, X, y)\|\}$ is bounded, and hence so is $\{\|T_y\|: y>0\}$, where $T_y=T(A, B, Q, y)$.

Conversely, if $\{\|T_y\|: y>0\}$ is bounded there is a sequence $\{T_{y_n}\}$ with $y_n\to 0$ that converges weakly to a bounded operator T. Lemma 3 assures us that $\langle Tf, Bg\rangle-\langle TAf, g\rangle=\langle Qf, g\rangle$ for all $f\in\mathfrak{D}(A)$ and $g\in\mathfrak{D}(B)$, so T satisfies ($\mathbf{E}'$), which in turn implies that T is a solution of ($\mathbf{E}$).

The proof of Theorem 1 can be easily modified to prove a more general result. Suppose $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are separable complex Hilbert spaces with B selfadjoint on $\mathfrak{H}_1$ and A selfadjoint on $\mathfrak{H}_2$. Let Q be a bounded operator on $\mathfrak{H}_2$ into $\mathfrak{H}_1$. Consider the operator equation

($\mathbf{F}$) $BXf-XAf=Qf$ for all f in $\mathfrak{D}(A)$, where X is a bounded operator on $\mathfrak{H}_2$ into $\mathfrak{H}_1$.

Define T_y by (3).

THEOREM 2. *Theorem* 1 *holds with* '**E**' *replaced by* '**F**'.

We can simplify the expression for T_y in case Q has a finite dimensional range.

COROLLARY. *Suppose A and B have resolutions of the identity (E) and (F) respectively, and $Q=\sum_{j=1}^{n}\langle\cdot, \phi_j\rangle\psi_j$, where $\psi_j\in\mathfrak{H}_1$, $\phi_j\in\mathfrak{H}_2$, $j=1, \cdots, n$. Then* (**F**) *has a solution if and only if there exists a constant M such that*

$$\left|\sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(t-s+iy)^{-1}d_t\langle E_tf, \phi_j\rangle d_s\langle F_s\psi_j, g\rangle\right| \leqq M\|f\|\,\|g\|$$

for all $f\in\mathfrak{H}_2$, $g\in\mathfrak{H}_1$, and $y>0$.

PROOF. This follows from the definition of T_y and the definite integral $(1/2\pi i)\int_{-\infty}^{\infty}(t-z)^{-1}(s-z^*)^{-1}dx=(s-t+2iy)^{-1}$.

3. **A similarity theorem.** In what follows we shall be dealing with operator matrices as defined in Halmos [2, Chapter 7]. As a typographical convenience we shall denote the operator matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{by } [A, B, C, D].$$

We will be concerned with extensions to Hilbert space of the following theorem of W. E. Roth [8]:

THEOREM. *Suppose $\mathfrak{F}$ is a field and A, B, and Q are square matrices of order n with elements in $\mathfrak{F}$. Then the equation $BX-XA=Q$ has a solution X with elements in $\mathfrak{F}$ if and only if $[B, Q, 0, A]$ and $[B, 0, 0, A]$ are similar matrices.*

The 'only if' part of the result follows from the simple matrix identity

$$(5)\quad [I, -X, 0, I][B, 0, 0, A][I, X, 0, I] = [B, BX - XA, 0, A].$$

Thus, for example, if a, b and q are elements in a complex Banach algebra $\mathfrak{A}$ with identity and there exists an $x\in\mathfrak{A}$ such that $bx-xa=q$, then $[b, 0, 0, a]$ and $[b, q, 0, a]$, (and hence also $[a, 0, q, b]$), are similar on $\mathfrak{A}\oplus\mathfrak{A}$. Such an x exists, as noted in the introduction, if a and b have disjoint spectra. This was applied by Brown and Pearcy in [**1**].

The 'if' part of Roth's theorem provides the major challenge. B. B. Morrel [**5**] noted that the equation $bx-xa=q$ has a solution x in $\mathfrak{A}$ if and only if

$$[v_1, v_2, v_3, v_4][b, 0, 0, a] = [b, q, 0, a][v_1, v_2, v_3, v_4],$$

where $v_j\in\mathfrak{A}$, $j=1, 2, 3, 4$, and v_4 is invertible. We give an example to show that in general one cannot substitute invertibility of $[v_1, v_2, v_3, v_4]$ for invertibility of v_4 in the last statement. Let $\mathfrak{A}$ be the set of all bounded operators on an infinite dimensional Hilbert space and suppose that S is a nonunitary isometry on $\mathfrak{H}$, so $S^*S=I$ and $I-SS^*=P\neq 0$. Then $[S^*, O, P, S]^{-1}=[S, P, 0, S^*]$ is unitary on $\mathfrak{H}\oplus\mathfrak{H}$. Now,

$$[S^*, 0, P, S][S, 0, 0, 0][S, P, 0, S^*] = [S, P, 0, 0],$$

so $[S, 0, 0, 0]$ and $[S, P, 0, 0]$ are unitarily equivalent. But, suppose a bounded solution X exists to $SX=P$. Then $X=S^*SX=S^*P=0$, which is impossible.

We thus see that the conclusion of Roth's theorem is not in general valid for bounded operators A, B, Q on a Hilbert space. However, our last theorem shows that it is valid in case A and B are selfadjoint.

THEOREM 3. *Suppose A and B are bounded selfadjoint operators on complex separable Hilbert spaces $\mathfrak{H}_2$ and $\mathfrak{H}_1$ respectively. Then* (**F**) *has a solution if and only if $[B, Q, 0, A]$ and $[B, 0, 0, A]$ are similar operators on $\mathfrak{H}_1\oplus\mathfrak{H}_2$.*

PROOF. If (**F**) has a solution, then (5) shows that the operators $[B, Q, 0, A]$ and $[B, 0, 0, A]$ are similar.

Conversely suppose that these operators are similar, so there exists an operator V on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that $[B, Q, 0, A] = V^{-1}[B, 0, 0, A]V$. Then, for a complex $z = x + iy$ with $y > 0$,

$$[(B - zI)^{-1}, -(B - zI)^{-1}Q(A - zI)^{-1}, 0, (A - zI)^{-1}] = V^{-1}[(B - zI)^{-1}, 0, 0, (A - zI)^{-1}]V.$$

We act on the left of both sides of the above equation with

$$[I - 2iy(B - z^*I)^{-1}, 0, 0, I]$$

and obtain

$$\begin{aligned}[(B - z^*I)^{-1}, &-(B - z^*I)^{-1}Q(A - zI)^{-1}, 0, (A - zI)^{-1}] \\ &= V^{-1}[(B - zI)^{-1}, 0, 0, (A - zI)^{-1}]V \\ &\quad - 2iy[(B - z^*I)^{-1}, 0, 0, I]V^{-1}[(B - zI)^{-1}, 0, 0, (A - zI)^{-1}]V.\end{aligned}$$

Next apply $\lim_{\alpha \to \infty} (1/2\pi i)\int_{-\alpha}^{\alpha} \cdots dx$ to both sides. The left side becomes $[-1/2I, -T_y, 0, 1/2I]$ and by Lemmas 1 and 2, the entries on the right side are uniformly bounded for $y > 0$. Thus $\{\|T_y\| : y > 0\}$ is bounded and by Theorem 2 a solution X exists to **(F)**.

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