SUMS OF IRREDUCIBLE OPERATORS[1]

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Let $\mathcal{H}$ be a separable, infinite-dimensional complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A (closed, linear) subspace $\mathfrak{M}$ of $\mathcal{H}$ is said to *reduce* an operator $T\in\mathcal{B}(\mathcal{H})$ if $T\mathfrak{M}\subset\mathfrak{M}$ and $T\mathfrak{M}^\perp\subset\mathfrak{M}^\perp$. An operator is *irreducible* if the only subspaces which reduce it are $\{0\}$ and $\mathcal{H}$. Halmos has recently shown that the irreducible operators are dense in $\mathcal{B}(\mathcal{H})$ in the norm topology [**3**]. Our purpose here is to note that irreducible operators abound in another sense.

THEOREM 1. *Every operator in* $\mathcal{B}(\mathcal{H})$ *is the sum of four irreducible operators.*

This result will be proved by means of

THEOREM 2. *A selfadjoint operator is the real part of an irreducible operator if and only if it is not a scalar.*

We begin with the following lemma.

LEMMA. *Each projection other than* 0 *and* I *is the real part of an irreducible operator.*

Write $\mathcal{H}=\mathcal{K}\oplus\mathcal{K}$, let $\{e_n \mid n=1,2,\cdots\}$ and $\{f_n\mid n=1,2,\cdots\}$ be orthonormal bases of $\mathcal{K}$ with

$$f_1=\frac{\sqrt{6}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e_n,$$

and let S be the shift operator on $\mathcal{K}$ defined by $Sf_n=f_{n+1}$. With $A=S/\sqrt{2}$, the operator

$$U=\begin{pmatrix} A & (I-AA^*)^{1/2}\\ (I-A^*A)^{1/2} & -A^*\end{pmatrix}$$

on $\mathcal{H}$ is unitary [**2**, Problem 177], and

$$U\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}U^*=\frac{1}{2}\begin{pmatrix} SS^* & S\\ S^* & I\end{pmatrix}.$$

Therefore, the right side of this equation is a projection P of infinite rank and infinite nullity.

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Now let $K = A \oplus B$, where the selfadjoint operators A and B are defined on $\mathcal{K}$ by $Ae_n = e_n/n$ and $Be_n = e_n/n\pi$. We show that $P+iK$ is irreducible, which will prove the lemma for projections of infinite rank and nullity. To this end, let

$$\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

be a projection which commutes with $P+iK$, or equivalently with P and K. The commutativity with K gives $AX = XA$, $BZ = ZB$, and $AY = YB$, so $AYe_n = YBe_n = Ye_n/n\pi$, $Ye_n = 0$ for all n, and therefore $Y = 0$. Further, the equations $AX = XA$ and $BZ = ZB$ imply that X and Z are diagonal in the basis $\{e_n\}$ (that is, each e_n is an eigenvector for X and Z). Next, the commutativity with P gives $XSS^* = SS^*X$ and $XS = SZ$. Since SS^* is the projection on the span of the f_n for $n \geqq 2$, the first equation shows that f_1 is an eigenvector for X. Coupled with the fact that X is diagonal in the basis $\{e_n\}$, this shows that X is scalar. Then $XS = SZ$ implies $Z = S^*XS = XS^*S = X$. Consequently the projection

$$\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

is scalar, and must therefore be 0 or I.

We have not been able to find a proof of this type for projections of finite (or cofinite) rank. Before proceeding with this case, we wish to note that with the information at hand, we can prove a weakened form of Theorem 1, namely: every operator is the sum of a finite number of irreducible operators. Since every operator is a finite linear combination of projections [1], and since a projection is either the sum or the difference of two projections of infinite rank and nullity, it suffices to consider projections of infinite rank and nullity. For such a projection P, there is a selfadjoint operator K such that $P+iK$ is irreducible. Then $P = \frac{1}{2}(P+iK) + \frac{1}{2}(P-iK)$, and the operators $\frac{1}{2}(P \pm iK)$ are irreducible.

Returning to the proof of the lemma, let $\mathcal{K} = L^2(\mu)$, with μ Lebesgue measure on $[0, 1]$, and let P be the projection on the polynomials of degree at most $n-1$. Then P is of rank n, and we shall show that $P+iK$ is irreducible, where K is defined by $(Kf)(t) = tf(t)$ for all $f \in L^2$. If E is a projection which commutes with K, then E is given by multiplication by the characteristic function ϕ of a measurable subset A of $[0, 1]$ [2, Problem 115]. If E also commutes with P, then P leaves invariant the subspace of functions which vanish outside of A.

In particular $P\phi$ vanishes outside of A, and since it is a polynomial, we have either $\mu(A)=1$ or $P\phi=0$. In the former case $E=I$, and in the latter ϕ is orthogonal to the function identically equal to 1, so $\mu(A)=0$ and $E=0$. Therefore $P+iK$ is irreducible. Since $(I-P)+iK$ is also irreducible, this completes the proof of the lemma.

We remark that a proof along these lines may be constructed for the first part of the lemma. Let $\mathcal{H}=L^2(0, 2\pi)$, let $\{e_n\}$ be the usual basis of exponentials, let K be defined by $Ke_n=e_n/n$ for $n\neq 0$ and $Ke_0=0$, and let P be the projection on $L^2(0, 2\pi a)$ with a irrational and $a\in(0, 1)$. Then $P+iK$ is irreducible. The proof is similar to the above and will be omitted.

For the proof of Theorem 2, let H be a nonscalar selfadjoint operator. Then there exists a spectral projection P of H distinct from 0 and I. By the lemma there is a selfadjoint operator K such that $P+iK$ is irreducible. Since for any operator T, $TH=HT$ implies $TP=PT$, it follows that $H+iK$ is irreducible. The converse is obvious: an operator with scalar real part is normal and therefore reducible.

Finally we note that any selfadjoint operator is the sum of two irreducible operators, from which Theorem 1 follows. If H is nonscalar this is a consequence of Theorem 2. If $H=\alpha I$ and T is any irreducible operator, then $\alpha I-T$ is irreducible and $\alpha I=T+(\alpha I-T)$.

Added in proof. Heydar Radjavi has recently improved the number four in Theorem 1 to two, which is best possible. His result will appear in these Proceedings.

References

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2. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.

3. ———, *Irreducible operators*, Michigan Math. J. **15** (1968), 215–223.

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